Operator Theory:
Advances and Applications, Vol. 221, 71-108

Accretive (∗)-extensions and Realization Problems

Yury Arlinskiĭ, Sergey Belyi and Eduard Tsekanovskiĭ

Dedicated to Heinz Langer on the occasion of his 75^{th} *birthday*

Abstract. We present a solution of the extended Phillips-Kato extension problem about existence and parametrization of all accretive (∗)-extensions (with the exit into triplets of rigged Hilbert spaces) of a densely defined non-negative operator. In particular, the analogs of the von Neumann and Friedrichs theorems for existence of non-negative self-adjoint (∗)-extensions are obtained. Relying on these results we introduce the extremal classes of Stieltjes and inverse Stieltjes functions and show that each function from these classes can be realized as the impedance function of an L-system. It is proved that in this case the realizing L-system contains an accretive operator and, in case of Stieltjes functions, an accretive (∗)-extension. Moreover, we establish the connection between the above-mentioned classes and the Friedrichs and Kreĭn-von Neumann extremal non-negative extensions.

Mathematics Subject Classification (2000). Primary 47A10, 47B44; Secondary 46E20, 46F05.

Keywords. Accretive operator, L-system, realization.

1. Introduction

We recall that an operator-valued function $V(z)$ acting on a finite-dimensional Hilbert space E belongs to the class of operator-valued Herglotz-Nevanlinna functions if it is holomorphic on $\mathbb{C}\setminus\mathbb{R}$, if it is symmetric with respect to the real axis, i.e., $V(z)^* = V(\bar z)$, $z\in\mathbb{C}\setminus\mathbb{R}$, and if it satisfies the positivity condition

$$\operatorname{Im} V(z)\ge 0,\quad z\in\mathbb{C}_+.$$

It is well known (see, e.g., [14], [16]) that operator-valued Herglotz-Nevanlinna functions admit the following integral representation:

$$V(z)=Q+Lz+\int_{\mathbb{R}}\left(\frac{1}{t-z}-\frac{t}{1+t^2}\right)dG(t),\quad z\in\mathbb{C}\setminus\mathbb{R},\tag{1.1}$$

where $Q = Q^*$, $L \geq 0$, and $G(t)$ is a nondecreasing operator-valued function on $\mathbb{R}$ with values in the class of nonnegative operators in E such that

$$\int_{\mathbb{R}} \frac{(dG(t)x, x)_E}{1+t^2} < \infty, \quad x \in E. \tag{1.2}$$

The realization of a selected class of Herglotz-Nevanlinna functions is provided by a system Θ of the form

$$\begin{cases} (\mathbb{A} - zI)x = KJ\varphi_- \\ \varphi_+ = \varphi_- - 2iK^*x \end{cases} \tag{1.3}$$

or

$$\Theta = \begin{pmatrix} \mathbb{A} & K & J \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & E \end{pmatrix}. \tag{1.4}$$

In this system $\mathbb{A}$, the *state-space operator* of the system, is a so-called $(*)$-extension, which is a bounded linear operator from $\mathcal{H}_+$ into $\mathcal{H}_-$ extending a symmetric operator A in $\mathcal{H}$, where $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ is a rigged Hilbert space. Moreover, K is a bounded linear operator from the finite-dimensional Hilbert space E into $\mathcal{H}_-$, while $J = J^* = J^{-1}$ is acting on E, are such that $\operatorname{Im}\mathbb{A} = KJK^*$. Also, $\varphi_- \in E$ is an input vector, $\varphi_+ \in E$ is an output vector, and $x \in \mathcal{H}_+$ is a vector of the state space of the system Θ. The system described by (1.3)-(1.4) is called an *L-system.* The operator-valued function

$$W_\Theta(z) = I - 2iK^*(\mathbb{A} - zI)^{-1}KJ \tag{1.5}$$

is a transfer function of the system Θ. It was shown in [14] that an operator-valued function $V(z)$ acting on a Hilbert space E of the form (1.1) can be represented and realized in the form

$$V(z) = i[W_\Theta(z) + I]^{-1}[W_\Theta(z) - I] = K^*(\operatorname{Re}\mathbb{A} - zI)^{-1}K, \tag{1.6}$$

where $W_\Theta(z)$ is a transfer function of some canonical scattering ($J = I$) system Θ, and where the "*real part*" $\operatorname{Re}\mathbb{A} = \frac{1}{2}(\mathbb{A} + \mathbb{A}^*)$ of $\mathbb{A}$ satisfies $\operatorname{Re}\mathbb{A} \supset \hat{A} = \hat{A}^* \supset \dot{A}$ if and only if the function $V(z)$ in (1.1) satisfies the following two conditions:

$$\begin{cases} L = 0, \\ Qx = \int_{\mathbb{R}} \frac{t}{1+t^2}\, dG(t)x \quad \text{when} \quad \int_{\mathbb{R}} (dG(t)x, x)_E < \infty. \end{cases} \tag{1.7}$$

The class of all realizable Herglotz-Nevanlinna functions with conditions (1.7) is denoted by $N(R)$ (see [14]).

In the first part of this paper we present a solution of the extended Phillips-Kato extension problem. We show the existence and parameterize all accretive $(*)$-extensions (with the exit into triplets of rigged Hilbert spaces) of a densely defined non-negative symmetric operator. Moreover, the analogs of the von Neumann and Friedrichs theorems for existence of non-negative self-adjoint $(*)$-extensions are obtained. In the remaining part of the paper we focus on the introduced extremal classes of Stieltjes and inverse Stieltjes functions. We show that any function belonging to these classes can be realized as the impedance function of an L-system with special properties. In the end we establish the connection between the above-mentioned classes and the Friedrichs and Kreĭn-von Neumann extremal non-negative extensions.

The complete proofs of some parts of the material from [6], [20], [30] are presented here for the first time.

2. Preliminaries

For a pair of Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ we denote by $[\mathcal{H}_1, \mathcal{H}_2]$ the set of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. Let $\dot{A}$ be a closed, densely defined, symmetric operator in a Hilbert space $\mathcal{H}$ with inner product $(f, g), f, g \in \mathcal{H}$. Any operator T in $\mathcal{H}$ such that

$$\dot{A} \subset T \subset \dot{A}^*$$

is called a *quasi-self-adjoint extension* of $\dot{A}$.

Consider the rigged Hilbert space (see [14]) $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$, where $\mathcal{H}_+ = \mathrm{Dom}(\dot{A}^*)$ and

$$(f, g)_+ = (f, g) + (\dot{A}^* f, \dot{A}^* g), \quad f, g \in \mathrm{Dom}(A^*).$$

Let $\mathcal{R}$ be the *Riesz-Berezansky operator* $\mathcal{R}$ (see [14]) which maps $\mathcal{H}_-$ onto $\mathcal{H}_+$ such that $(f, g) = (f, \mathcal{R}g)_+$ $(\forall f \in \mathcal{H}_+,\ g \in \mathcal{H}_-)$ and $\|\mathcal{R}g\|_+ = \|g\|_-$. Note that identifying the space conjugate to $\mathcal{H}_\pm$ with $\mathcal{H}_\mp$, we get that if $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ then $\mathbb{A}^* \in [\mathcal{H}_+, \mathcal{H}_-]$.

Definition 2.1. An operator $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ is called a self-adjoint bi-extension of a symmetric operator $\dot{A}$ if $\mathbb{A} = \mathbb{A}^*$ and $\mathbb{A} \supset \dot{A}$.

Let $\mathbb{A}$ be a self-adjoint bi-extension of $\dot{A}$ and let the operator $\widehat{A}$ in $\mathcal{H}$ be defined as follows:

$$\mathrm{Dom}(\widehat{A}) = \{f \in \mathcal{H}_+ : \widehat{A}f \in \mathcal{H}\},\ \widehat{A} = \mathbb{A}\upharpoonright \mathrm{Dom}(\widehat{A}).$$

The operator $\widehat{A}$ is called a *quasi-kernel* of a self-adjoint bi-extension $\mathbb{A}$ (see [35]). We say that a self-adjoint bi-extension $\mathbb{A}$ of $\dot{A}$ is *twice-self-adjoint* or *t-self-adjoint* if its quasi-kernel $\widehat{A}$ is a self-adjoint operator in $\mathcal{H}$.

Definition 2.2. Let T be a quasi-self-adjoint extension of $\dot{A}$ with nonempty resolvent set $\rho(T)$. An operator $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ is called a (∗)-extension (or correct bi-extension) of an operator T if

1. $\mathbb{A} \supset T \supset \dot{A}, \quad \mathbb{A}^* \supset T^* \supset \dot{A}$,
2. the quasi-kernel of self-adjoint bi-extension $\mathrm{Re}\,\mathbb{A} = \frac{1}{2}(\mathbb{A} + \mathbb{A}^*)$ is a self-adjoint extension of $\dot{A}$.

The existence, description, and analog of von Neumann's formulas for self-adjoint bi-extensions and (∗)-extensions were discussed in [35] (see also [5], [9], [14]). In what follows we suppose that $\dot{A}$ has equal deficiency indices and will say that a quasi-self-adjoint extension T of $\dot{A}$ belongs to the **class** $\Lambda(\dot{A})$ if $\rho(T) \neq \emptyset$, $\mathrm{Dom}(\dot{A}) = \mathrm{Dom}(T) \cap \mathrm{Dom}(T^*)$, and T admits (∗)-extensions.

Recall that two quasi-self-adjoint extensions T_1 and T_2 of $\dot{A}$ are called **disjoint** if

$$\mathrm{Dom}(T_1) \cap \mathrm{Dom}(T_2) = \mathrm{Dom}(\dot{A})$$

and **transversal** if, in addition,

$$\mathrm{Dom}(T_1) + \mathrm{Dom}(T_2) = \mathrm{Dom}(\dot{A}^*).$$

Note that from von Neumann formulas immediately follows that two transversal self-adjoint extensions are automatically disjoint.

Let $\dot{A}$ be a closed densely defined symmetric operator and let $T \in \Lambda(\dot{A})$. It has been shown in [4] that $T \in \Lambda(\dot{A})$ if and only if there exists a self-adjoint extension $\tilde{A}$ of $\dot{A}$ transversal to T, and, moreover, the formulas

$$\mathbb{A} = \dot{A}^* - \mathcal{R}^{-1}\dot{A}^*(I - \mathcal{P}_{T\tilde{A}}), \quad \mathbb{A}^* = \dot{A}^* - \mathcal{R}^{-1}\dot{A}^*(I - \mathcal{P}_{T^*\tilde{A}}), \tag{2.1}$$

set a bijection between the set of all $(*)$-extensions of $T \in \Lambda(\dot{A})$ and their adjoint and the set of all all self-adjoint extensions $\tilde{A}$ of the operator $\dot{A}$ that are transversal to T. Here $\mathcal{P}_{T\tilde{A}}$ and $\mathcal{P}_{T^*\tilde{A}}$ are the projectors in $\mathcal{H}_+$ onto $\mathrm{Dom}(T)$ and $\mathrm{Dom}(T^*)$, corresponding to the direct decompositions

$$\mathcal{H}_+ = \mathrm{Dom}(T)\dot{+}\mathfrak{M}_{\tilde{A}}, \quad \mathcal{H}_+ = \mathrm{Dom}(T^*)\dot{+}\mathfrak{M}_{\tilde{A}}, \tag{2.2}$$

where $\mathfrak{M}_{\tilde{A}} = \mathrm{Dom}(\tilde{A}) \ominus \mathrm{Dom}(\dot{A})$. If a $(*)$-extension $\mathbb{A}$ of T takes the form (2.1), we say that $\mathbb{A}$ is **generated** by $\tilde{A}$.

It is shown in [4] that if deficiency indices of $\dot{A}$ are finite and equal, then for each quasi-self-adjoint extension T of $\dot{A}$ with $\rho(T) \neq \emptyset$ there exists a self-adjoint extension of $\dot{A}$ transversal to T. The latter is also true if there is $z \in \mathbb{C}$ such that $z, \bar{z} \in \rho(T)$ even for the case of infinite deficiency indices of $\dot{A}$.

Recall that a linear operator T in a Hilbert space $\mathfrak{H}$ is called **accretive** [24] if $\mathrm{Re}\,(Tf, f) \geq 0$ for all $f \in \mathrm{Dom}(T)$ and **maximal accretive** (m-accretive) if it is accretive and has no accretive extensions in $\mathfrak{H}$. The following statements are equivalent [31]:

(i) the operator T is m-accretive;
(ii) the operator T is accretive and its resolvent set contains points from the left half-plane;
(iii) the operators T and T^* are accretive.

The resolvent set $\rho(T)$ of m-accretive operator contains the open left half-plane Π_- and

$$||(T - zI)^{-1}|| \leq \frac{1}{|\mathrm{Re}\, z|}\,, \ \mathrm{Re}\, z < 0.$$

Let $\mathcal{A}$ and $\mathcal{B}$ be two densely defined closed accretive operators such that

$$(\mathcal{A}f, g) = (f, \mathcal{B}g), \ f \in \mathrm{Dom}(\mathcal{A}), \ g \in \mathrm{Dom}(\mathcal{B}).$$

It was proved in [31] that there exists a maximal accretive operator T such that

$$T \supset \mathcal{A} \quad \text{and} \quad T^* \supset \mathcal{B}.$$

In particular, it follows that if $\dot{A}$ is nonnegative symmetric operator, then there exist maximal accretive quasi-self-adjoint extensions of $\dot{A}$.

Let T be a quasi-self-adjoint maximal accretive extension of a nonnegative operator $\dot{A}$. A $(*)$-extension $\mathbb{A}$ of T is called accretive if $\mathrm{Re}\,(\mathbb{A}f,f)\geq 0$ for all $f\in\mathcal{H}_+$. This is equivalent to that the real part $\mathrm{Re}\,\mathbb{A}=(\mathbb{A}+\mathbb{A}^*)/2$ is nonnegative self-adjoint bi-extension of $\dot{A}$.

Definition 2.3. Let $\dot{A}$ have finite equal deficiency indices. A system of equations

$$\begin{cases} (\mathbb{A}-zI)x=KJ\varphi_- \\ \varphi_+=\varphi_- -2iK^*x \end{cases},$$

or an array

$$\Theta=\begin{pmatrix} \mathbb{A} & K & J \\ \mathcal{H}_+\subset\mathcal{H}\subset\mathcal{H}_- & & E \end{pmatrix} \tag{2.3}$$

is called an **L-system** if:

(1) $\mathbb{A}$ is a $(*)$-extension of an operator T of the class $\Lambda(\dot{A})$;
(2) $J=J^*=J^{-1}\in[E,E]$, $\quad\dim E<\infty$;
(3) $\mathrm{Im}\,\mathbb{A}=KJK^*$, where $K\in[E,\mathcal{H}_-]$, $K^*\in[\mathcal{H}_+,E]$, and

$$\mathrm{Ran}(K)=\mathrm{Ran}(\mathrm{Im}\,\mathbb{A}). \tag{2.4}$$

In the definition above $\varphi_-\in E$ stands for an input vector, $\varphi_+\in E$ is an output vector, and x is a state space vector in $\mathcal{H}$. An operator $\mathbb{A}$ is called a *state-space operator* of the system Θ, J is a *direction operator*, and K is a *channel operator*. A system Θ of the form (2.3) is called an *accretive system* [17] if its main operator $\mathbb{A}$ is accretive and *accumulative system* [18] if its main operator $\mathbb{A}$ is accumulative, i.e., satisfies

$$(\mathrm{Re}\,\mathbb{A}f,f)\leq(\dot{A}^*f,f)+(f,\dot{A}^*f),\quad f\in\mathcal{H}_+. \tag{2.5}$$

We associate with an L-system Θ the operator-valued function

$$W_\Theta(z)=I-2iK^*(\mathbb{A}-zI)^{-1}KJ,\quad z\in\rho(T), \tag{2.6}$$

which is called a **transfer operator-valued function** of the L-system Θ. We also consider the operator-valued function

$$V_\Theta(z)=K^*(\mathrm{Re}\,\mathbb{A}-zI)^{-1}K. \tag{2.7}$$

It was shown in [14] that both (2.6) and (2.7) are well defined. The transfer operator-function $W_\Theta(z)$ of the system Θ and an operator-function $V_\Theta(z)$ of the form (2.7) are connected by the relations valid for $\mathrm{Im}\,z\neq 0$, $z\in\rho(T)$,

$$\begin{aligned} V_\Theta(z)&=i[W_\Theta(z)+I]^{-1}[W_\Theta(z)-I]J,\\ W_\Theta(z)&=(I+iV_\Theta(z)J)^{-1}(I-iV_\Theta(z)J). \end{aligned} \tag{2.8}$$

The function $V_\Theta(z)$ defined by (2.7) is called the **impedance function** of an L-system Θ of the form (2.3). It was shown in [14] that the class $N(R)$ of all Herglotz-Nevanlinna functions in a finite-dimensional Hilbert space E that can be realized as impedance functions of an L-system is described by conditions (1.7). In particular, the following theorem [3], [14] takes place.

Theorem 2.4. *Let Θ be an L-system of the form* (2.3)*. Then the impedance function $V_\Theta(z)$ of the form* (2.7) *belongs to the class $N(R)$.*

Conversely, let an operator-valued function $V(z)$ belong to the class $N(R)$. Then $V(z)$ can be realized as the impedance function of an L-system Θ of the form (2.3) *with a preassigned direction operator J for which $I + iV(-i)J$ is invertible.*

We will heavily rely on Theorem 2.4 in the last two sections of the present paper.

3. The Friedrichs and Kreĭn-von Neumann extensions

We recall that a symmetric operator $\dot{B}$ is called **non-negative** if

$$(\dot{B}f, f) \geq 0, \quad \forall f \in \mathrm{Dom}(\dot{B}).$$

Let $\dot{B}$ be a closed densely defined non-negative operator a Hilbert space $\mathcal{H}$ and let $\dot{B}^*$ be its adjoint. Consider the sesquilinear form $\tau_{\dot{B}}[f,g] = (\dot{B}f, g)$, $f, g \in \mathrm{Dom}(\dot{B})$. A sequence $\{f_n\} \subset \mathrm{Dom}(\dot{B})$ is called $\tau_{\dot{B}}$**-converging** to the vector $u \in \mathcal{H}$ if

$$\lim_{n\to\infty} f_n = u \quad \text{and} \quad \lim_{n,m\to\infty} \tau_{\dot{B}}[f_n - f_m] = 0.$$

The form $\tau_{\dot{B}}$ is **closable** [24], i.e., there exists a minimal closed extension (the closure) of $\tau_{\dot{B}}$. Following the M. Kreĭn notations we denote by $\dot{B}[\cdot,\cdot]$ the closure of $\tau_{\dot{B}}$ and by $\mathcal{D}[\dot{B}]$ its domain. By definition $\dot{B}[u] = \dot{B}[u,u]$ for all $u \in \mathcal{D}[\dot{B}]$. Because $\dot{B}[u,v]$ is closed, it possesses the property: if

$$\lim_{n\to\infty} u_n = u \quad \text{and} \quad \lim_{n,m\to\infty} \dot{B}[u_n - u_m] = 0,$$

then $\lim\limits_{n\to\infty} \dot{B}[u - u_n] = 0$. The **Friedrichs extension** B_F of $\dot{B}$ is defined as a non-negative self-adjoint operator associated with the form $\dot{B}[\cdot,\cdot]$ by the First Representation Theorem [24]:

$$(B_F u, v) = \dot{B}[u,v] \quad \text{for all} \quad u \in \mathrm{Dom}(B_F) \quad \text{and for all} \quad v \in \mathcal{D}[\dot{B}].$$

It follows that

$$\mathrm{Dom}(B_F) = \mathcal{D}[\dot{B}] \cap \mathrm{Dom}(\dot{B}^*),\ B_F = \dot{B}^* \upharpoonright \mathrm{Dom}(B_F).$$

The Friedrichs extension B_F is a unique non-negative self-adjoint extension having the domain in $\mathcal{D}[\dot{B}]$. Notice that by the Second Representation Theorem [24] one has

$$\mathcal{D}[\dot{B}] = \mathcal{D}[B_F] = \mathrm{Dom}(B_F^{1/2}),\ \dot{B}[u,v] = (B_F^{1/2}u, B_F^{1/2}v),\ u, v \in \mathcal{D}[\dot{B}].$$

Let $\dot{B}$ be a non-negative closed densely defined symmetric operator. Consider the family of symmetric contractions

$$\dot{A}^{(a)} = (aI - \dot{B})(aI + \dot{B})^{-1},\ a > 0,$$

defined on $\mathrm{Dom}(\dot{A}^{(a)}) = (aI+\dot{B})\mathrm{Dom}(\dot{B})$. Notice that the orthogonal complement $\mathfrak{N}^{(a)} = \mathcal{H} \ominus \mathrm{Dom}(\dot{A}^{(a)})$ coincides with the defect subspace $\mathfrak{N}_{-a}$ of the operator $\dot{B}$. Let $\dot{A} = \dot{A}^{(1)}$ and let $b = (1-a)(a+1)^{-1}$. Then $b \in (-1, 1)$ and

$$\dot{A}^{(a)} = (\dot{A} - bI_{\mathcal{H}})(I - b\dot{A})^{-1}.$$

Clearly, there is a one-one correspondence given by the Cayley transform

$$B = a(I - A^{(a)})(I + A^{(a)})^{-1}, \quad A^{(a)} = (aI - B)(aI + B)^{-1},$$

between all non-negative self-adjoint extensions B of the operator $\dot{B}$ and all self-adjoint contractive (sc) extensions $A^{(a)}$ of $\dot{A}^{(a)}$. As was established by M. Kreĭn in [25], [26] the set of all sc-extensions of $\dot{A}$ forms an operator interval $[A_\mu, A_M]$. Following M. Kreĭn's notations we call the extreme contractive self-adjoint extensions A_μ and A_M of a symmetric contraction $\dot{A}$ by the **rigid** and the **soft** extensions, respectively. The next result describe the sesquilinear form $B[u, v]$ by means the fractional-linear transformation $A = (I - B)(I + B)^{-1}$.

Proposition 3.1.

(1) *Let B be a non-negative self-adjoint operator and let $A = (I - B)(I + B)^{-1}$ be its Cayley transform. Then*

$$\begin{aligned}&\mathcal{D}[B] = \mathrm{Ran}((I + A)^{1/2}),\\ &B[u, v] = -(u, v) + 2\left((I + A)^{-1/2}u, (I + A)^{-1/2}v\right), \quad u, v \in \mathcal{D}[B].\end{aligned} \tag{3.1}$$

(2) *Let $\dot{B}$ be a closed densely defined non-negative symmetric operator and let B be its non-negative self-adjoint extension. If $\dot{A} = (I - \dot{B})(I + \dot{B})^{-1}$, $A = (I - B)(I + B)^{-1}$, then*

$$\mathcal{D}[B] = \mathrm{Ran}(I + A_\mu)^{1/2} \dotplus \mathrm{Ran}(A - A_\mu)^{1/2}. \tag{3.2}$$

Proof. (1). Since $B = (I - A)(I + A)^{-1}$, one obtains with $f = (I + A)h$,

$$\begin{aligned}B[f] = ((I - A)h, (I + A)h) &= -\|(I + A)h\|^2 + 2\|(I + A)^{1/2}h\|^2\\ &= -\|f\|^2 + 2\|(I + A)^{-1/2}f\|^2.\end{aligned}$$

Now the closure procedure leads to (3.1).

(2) Since A is a sc-extension of $\dot{A}$, we get $A_\mu \le A \le A_M$. Hence $I + A = I + A_\mu + (A - A_\mu)$. Because $I + A_\mu$ and $A - A_\mu$ are non-negative self-adjoint operators, we get the equality [21]:

$$\mathrm{Ran}((I + A)^{1/2}) = \mathrm{Ran}((I + A_\mu)^{1/2}) + \mathrm{Ran}((A - A_\mu)^{1/2}).$$

Since $\mathrm{Ran}((I+A_\mu)^{1/2}) \cap \mathfrak{N} = \{0\}$, where $\mathfrak{N} = \mathcal{H} \ominus \mathrm{Dom}(\dot{A})$, and $\mathrm{Ran}(A-A_\mu) \subseteq \mathfrak{N}$, we get $\mathrm{Ran}((I + A_\mu)^{1/2}) \cap \mathrm{Ran}((A - A_\mu)^{1/2}) = \{0\}$. Then we arrive to (3.2). □

We note that $\mathrm{Ran}(\tilde{B}^{1/2}) = \mathrm{Ran}((I - \tilde{A})^{1/2})$. Now let A_μ and A_M be the rigid and the soft extensions of $\dot{A}$. Then the operators

$$B_F = (I - A_\mu)(I + A_\mu)^{-1}, \tag{3.3}$$

and

$$B_K = (I - A_M)(I + A_M)^{-1}, \tag{3.4}$$

are non-negative self-adjoint extensions of $\dot{B}$. It also follows (see [25], [26]) that

$$B_F = a(I - A_\mu^{(a)})(I + A_\mu^{(a)})^{-1}, \quad B_K = a(I - A_M^{(a)})(I + A_M^{(a)})^{-1}.$$

Since, the operators $A_\mu^{(a)}$ and $A_M^{(a)}$ posses the properties

$$\mathrm{Ran}((I + A_\mu^{(a)})^{1/2}) \cap \mathfrak{N}_{-a} = \mathrm{Ran}((I - A_M^{(a)})^{1/2}) \cap \mathfrak{N}_{-a} = \{0\},$$

we get the following result [25].

Proposition 3.2. *Let B be a non-negative self-adjoint extension of $\dot{B}$ and let $E(\lambda)$ be its resolution of identity. Then*

1. *$B = B_F$ if and only if at least for one $a > 0$ (then for all $a > 0$) the relation*
$$\int_0^\infty \lambda(dE(\lambda)\varphi, \varphi) = +\infty, \tag{3.5}$$
holds for each $\varphi \in \mathfrak{N}_{-a} \setminus \{0\}$;
2. *$B = B_K$ if and only if at least for one $a > 0$ (then for all $a > 0$) the relation*
$$\int_0^\infty \frac{(dE(\lambda)\varphi, \varphi)}{\lambda} = +\infty, \tag{3.6}$$
holds for each $\varphi \in \mathfrak{N}_{-a} \setminus \{0\}$.

The self-adjoint extension B_F given by (3.3) coincides [25] with the *Friedrichs extension* of $\dot{B}$. In the sequel we will call the operator B_K defined in (3.4) by the **Kreĭn-von Neumann extension** of $\dot{B}$.

4. Bi-extensions of non-negative symmetric operators

First we consider the case of bounded non-densely defined non-negative symmetric operator $\dot{B}$.

Theorem 4.1. *Let $\dot{B}$ be a bounded non-densely defined non-negative symmetric operator in a Hilbert space $\mathcal{H}$, $\mathrm{Dom}(\dot{B}) = \mathcal{H}_0$. Let $\dot{B}^* \in [\mathcal{H}, \mathcal{H}_0]$ be the adjoint of $\dot{B}$. Put $\dot{B}_0 = P_{\mathcal{H}_0}\dot{B}$, $\mathfrak{L} = \mathcal{H} \ominus \mathcal{H}_0$, where $P_{\mathcal{H}_0}$ is an orthogonal projection in $\mathcal{H}$ onto $\mathcal{H}_0$. Then the following statements are equivalent:*

(i) *$\dot{B}$ admits bounded non-negative self-adjoint extensions in $\mathcal{H}$;*

(ii) $\sup\limits_{f \in \mathcal{H}_0} \dfrac{||\dot{B}f||^2}{(\dot{B}f, f)} < \infty$;

(iii) $\dot{B}^*\mathfrak{L} \subseteq \mathrm{Ran}(\dot{B}_0^{1/2})$.

Proof. Since $(\dot{B}f, f) = ||\dot{B}_0^{1/2} f||^2$, $f \in \mathcal{H}_0$, and

$$\dot{B}^* = \dot{B}_0 P_{\mathcal{H}_0} + \dot{B}^* P_{\mathfrak{L}},$$

conditions (i) and (ii) are equivalent due to the Douglas Theorem [19]. Suppose $\dot{B}$ admits a bounded non-negative self-adjoint extension B. Then for $f \in \mathcal{H}_0$ one has

$$\begin{aligned}||\dot{B}f||^2 = ||Bf||^2 = ||B^{1/2}B^{1/2}f||^2 \le ||B^{1/2}||^2||B^{1/2}f||^2 \\ = ||B^{1/2}||^2(Bf,f) = ||B^{1/2}||^2(\dot{B}f,f) = ||B^{1/2}||^2||\dot{B}_0^{1/2}f||^2.\end{aligned}$$

It follows that statement (ii) holds true.

Now suppose that (iii) is fulfilled. Then the operator $L_0 := \dot{B}_0^{[-1/2]}\dot{B}^*\restriction\mathfrak{L}$ is bounded, where $\dot{B}_0^{[-1/2]}$ is the Moore-Penrose inverse to $\dot{B}_0^{1/2}$. Let $L_0^* \in [\mathcal{H}_0, \mathfrak{L}]$ be the adjoint to L_0. Set

$$\mathcal{B}_0 = \dot{B}P_{\mathcal{H}_0} + (\dot{B}^* + L_0^*L_0)P_{\mathfrak{L}}. \tag{4.1}$$

Then $\mathcal{B}_0$ is bounded extension of $\dot{B}$ in $\mathcal{H}$. Let $P_{\mathfrak{L}}$ be the orthogonal projection operator in $\mathcal{H}$ onto $\mathfrak{L}$. For $h \in \mathcal{H}$ we have

$$\begin{aligned}(\mathcal{B}_0h,h) &= (\dot{B}P_{\mathcal{H}_0}h + (\dot{B}^* + L_0^*L_0)P_{\mathfrak{L}}h, P_{\mathcal{H}_0}h + P_{\mathfrak{L}}h)\\ &= ||\dot{B}_0^{1/2}P_{\mathcal{H}_0}h||^2 + ||L_0P_{\mathfrak{L}}h||^2 + 2\mathrm{Re}\,(P_{\mathcal{H}_0}h, \dot{B}^*P_{\mathfrak{L}}h)\\ &= ||\dot{B}_0^{1/2}P_{\mathcal{H}_0}h||^2 + ||L_0P_{\mathfrak{L}}h||^2 + 2\mathrm{Re}\,(\dot{B}_0^{1/2}P_{\mathcal{H}_0}h, \dot{B}_0^{[-1/2]}\dot{B}^*P_{\mathfrak{L}}h)\\ &= ||\dot{B}_0^{1/2}P_{\mathcal{H}_0}h + L_0P_{\mathfrak{L}}h||^2.\end{aligned}$$

Thus, $\mathcal{B}_0$ is non-negative bounded self-adjoint extension of $\dot{B}$. Therefore (i) is equivalent to (iii). □

Remark 4.2. It is easy to see that the conditions

1. $\sup\limits_{f\in\mathcal{H}_0}\dfrac{||\dot{B}f||^2}{(\dot{B}f,f)} < \infty$,
2. there exists $c > 0$ such that $|(\dot{B}f,g)|^2 \le c(\dot{B}f,f)||g||^2,\ f \in \mathcal{H}_0, g \in \mathcal{H}$,
3. there exists $c > 0$ such that $|(\dot{B}f,g)|^2 \le c(\dot{B}f,f)||g||^2,\ f \in \mathcal{H}_0, g \in \mathfrak{L}$

are equivalent.

Now we consider semi-bounded (in particular non-negative) symmetric densely defined operators $\dot{A}$,

$$(\dot{A}x,x) \ge m(x,x), \qquad x \in \mathrm{Dom}(\dot{A}).$$

According to the classical von Neumann's theorem there exists a self-adjoint extension A of $\dot{A}$ with an arbitrary close to m lower bound. It was shown later by Friedrichs that operator $\dot{A}$ actually admits a self-adjoint extension with the same lower bound. In this section we are going to show that for the case of a self-adjoint bi-extension of $\dot{A}$ the analogue of von Neumann's theorem is true while the analogue of the Friedrichs theorem, generally speaking, does not take place.

Theorem 4.3. *Let $\dot{A}$ be a semi-bounded operator with a lower bound m and $\hat{A}$ be its symmetric extension with the same lower bound. Then $\dot{A}$ admits a self-adjoint* *bi-extension $\mathbb{A}$ with the same lower bound and containing $\hat{A}$ ($\mathbb{A} \supset \hat{A}$) if and only if there exists a number $k > 0$ such that*

$$\left|((\hat{A} - mI)f, h)\right|^2 \leq k((\hat{A} - mI)f, f)\,\|h\|_+^2, \tag{4.2}$$

for all $f \in \mathrm{Dom}(\hat{A})$, $h \in \mathcal{H}_+$.

Proof. Let $\mathcal{H}_+ \subseteq \mathcal{H} \subseteq \mathcal{H}_-$ be the rigged triplet generated by $\dot{A}$ and $\mathcal{R}$ be a Riesz-Berezansky operator corresponding to this triplet. In the Hilbert space $\mathcal{H}_+$ consider the operator

$$\dot{B} := \mathcal{R}(\hat{A} - mI), \quad \mathrm{Dom}(\dot{B}) = \mathrm{Dom}(\hat{A}).$$

Then $(\dot{B}f, f)_+ = ((\hat{A}f - mI)f, f) \geq 0$ for all $f \in \mathrm{Dom}(\dot{B})$. Observe that $\mathbb{A}$ is a self-adjoint bi-extension of $\dot{A}$ containing $\hat{A}$ if and only if the operator $B := \mathcal{R}\mathbb{A}$ is a $(+)$-bounded and $(+)$-self-adjoint extension of the operator $\dot{B}$ in $\mathcal{H}_+$. It follows from Theorem 4.1 and Remark 4.2 that the operator $\dot{B}$ admits $(+)$-non-negative bounded self-adjoint extension in $\mathcal{H}_+$ if and only if there exists $k > 0$ such that

$$|(\dot{B}f, h)_+|^2 \leq k(\dot{B}f, f)_+ ||h||_+^2, \ f \in \mathrm{Dom}(\dot{B}), h \in \mathcal{H}_+.$$

This is equivalent to (4.2) □

Remark 4.2 yields that if $\dot{A}$ has at least one self-adjoint bi-extension $\mathbb{A}$ containing $\hat{A}$ with the same lower bound, then it has infinitely many of such bi-extensions.

Corollary 4.4. *Inequalities* (4.2) *take place if and only if there exists a constant $C > 0$ such that*

$$|((\hat{A} - mI)f, \varphi_a)|^2 \leq C((\hat{A} - mI)f, f)\,\|\varphi_a\|_+^2, \tag{4.3}$$

for all $f \in \mathrm{Dom}(\hat{A})$ and all φ_a such that $(\dot{A}^* - (m - a)I)\varphi_a = 0$, ($a > 0$).

Proof. Suppose (4.2). Then for $h = \varphi_a \in \ker(\dot{A}^* - (m - a)I))$ we have (4.3). Now let us show that (4.2) follows from (4.3). It is known that there exists a self-adjoint extension A of $\hat{A}$ (for instance, the Friedrichs extension of $\hat{A}$) with the lower bound m. If λ is a regular point for A, then

$$\mathcal{H}_+ = \mathrm{Dom}(A) \dotplus \mathfrak{N}_\lambda. \tag{4.4}$$

Indeed, if $f \in \mathcal{H}_+ = \mathrm{Dom}(\dot{A}^*)$, then there exists an element $g \in \mathrm{Dom}(A)$ such that $(\dot{A}^* - \lambda I)f = (A - \lambda I)g$. This implies $(\dot{A}^* - \lambda I)(f - g) = 0$ and hence $(f - g) \in \mathfrak{N}_\lambda$ for any $f \in \mathcal{H}_+$ and $g \in \mathrm{Dom}(A)$, which confirms (4.4). Further, applying Cauchy-Schwartz inequality we obtain

$$\begin{aligned} |((\hat{A} - mI)f, g)|^2 &\leq ((\hat{A} - mI)f, f)((A - mI)g, g) \\ &\leq \hat{C}((\hat{A} - mI)f, f)\|g\|_+^2, \end{aligned} \tag{4.5}$$

for $f \in \mathrm{Dom}(\hat{A})$ and $g \in \mathrm{Dom}(A)$. Clearly, all the points of the form $(m - a)$, $(a > 0)$ are regular points for A and the points of a regular type for $\dot{A}$. Thus (4.4) implies

$$\mathcal{H}_+ = \mathrm{Dom}(A) \dot{+} \mathfrak{N}_{m-a}. \tag{4.6}$$

Let $h \in \mathcal{H}_+$ be an arbitrary vector. Applying (4.6) we get $h = g + \psi_a$, where $g \in \mathrm{Dom}(A)$ and $\psi_a \in \mathfrak{N}_{m-a}$. Adding up inequalities (4.3) and (4.5) and taking into account that the norms $\|\cdot\|$ and $\|\cdot\|_+$ are equivalent on $\mathfrak{N}_{m-a}$ we get (4.2). □

The following theorem is the analogue of the classical von Neumann's result.

Theorem 4.5. *Let ε be an arbitrary small positive number and $\dot{A}$ be a semi-bounded operator with the lower bound m. Then there exist infinitely many semi-bounded self-adjoint bi-extensions with the lower bound $(m-\varepsilon)$.*

Proof. First we show that the inequality

$$|((\dot{A}-(m-\varepsilon)I)f,g)|^2 \leq k((\dot{A}-(m-\varepsilon)I)f,f)\|g\|_+^2,$$

takes place for all $f \in \mathrm{Dom}(\dot{A})$, $g \in \mathfrak{M}$, and $k>0$. Indeed,

$$\begin{aligned}
|((\dot{A}-(m-\varepsilon)I)f,g)| &= |(f,(\dot{A}^*-(m-\varepsilon I)g)| \\
&\leq |(f,\dot{A}^*g)| + |m-\varepsilon|\cdot|(f,g)| \leq \|f\|\cdot\|A^*g\| + |m-\varepsilon|\cdot\|f\|\cdot\|g\| \\
&\leq \frac{1}{\sqrt{\varepsilon}}((\dot{A}-(m-\varepsilon)I)f,f)^{1/2}\|g\|_+ + \frac{|m-\varepsilon|}{\sqrt{\varepsilon}}((\dot{A}-(m-\varepsilon)I)f,f)^{1/2}\|g\|_+ \\
&= \frac{1+|m-\varepsilon|}{\sqrt{\varepsilon}}((\dot{A}-(m-\varepsilon)I)f,f)\|g\|_+.
\end{aligned}$$

The statement of the theorem follows from Theorem 4.3 and Remark 4.2. □

Theorem 4.6. *A non-negative densely-defined operator $\dot{A}$ admits a non-negative self-adjoint bi-extension if and only if the Friedrichs and Kreĭn-von Neumann extensions of $\dot{A}$ are transversal.*

Proof. It was shown in [28] (see also [12]) that the Friedrichs and Kreĭn-von Neumann extensions are transversal if and only if

$$\mathrm{Dom}(\dot{A}^*) \subseteq \mathcal{D}[A_K]. \tag{4.7}$$

Suppose that the Friedrichs extension A_F and the Kreĭn-von Neumann extension A_K of the operator $\dot{A}$ are transversal. Then the inclusion (4.7) holds. This means that $\mathcal{H}_+ \subseteq \mathrm{Dom}(A_K^{1/2})$. Since $||h||_+ \geq ||h||$ for all $h \in \mathcal{H}_+$, and $A_K^{1/2}$ is closed in $\mathcal{H}$, the closed graph theorem yields now that $A_K^{1/2} \in [\mathcal{H}_+, \mathcal{H}]$, i.e., there exists a number $c>0$ such that

$$||A_K^{1/2}u||^2 = A_K[u] \leq c||u||_+^2.$$

It follows that the sesquilinear form $A_K[u,v] = (A_K^{1/2}u, A_K^{1/2}v)$, $u,v \in \mathcal{H}_+$ is bounded on $\mathcal{H}_+$. Therefore, by Riesz theorem, there exists an operator $\mathbb{A}_K \in [\mathcal{H}_+, \mathcal{H}_-]$ such that

$$(\mathbb{A}_K u, v) = A_K[u,v], \quad u,v \in \mathcal{H}_+,\ u \in \mathcal{H}_+.$$

Due to $A_K[u] \geq 0$ for all $u \in \mathcal{D}[A_K]$, the operator $\mathbb{A}_K$ is non-negative. Since $(A_K u, v) = A_K[u, v]$ for all $u \in \mathrm{Dom}(A_K)$ and all $v \in \mathcal{D}[A_K]$, we get

$$(\mathbb{A}_K u, v) = (A_K u, v), \quad u \in \mathrm{Dom}(A_K),\ v \in \mathcal{H}_+.$$

Hence $\mathbb{A}_K \supset A_K$, i.e., $\mathbb{A}_K$ is t-self-adjoint bi-extension of $\dot{A}$ with quasi-kernel A_K.

Conversely, let $\dot{A}$ admits a non-negative self-adjoint bi-extension. Then from Theorem 4.3 we get the equality

$$|(\dot{A}f, h)|^2 \leq k(\dot{A}f, f)||h||_+^2,$$

for all $f \in \mathrm{Dom}(\dot{A})$ and all $h \in \mathcal{H}_+ = \mathrm{Dom}(\dot{A}^*)$, and some $k > 0$. Applying the theorem by T. Ando and K. Nishio [2] (see also [12]) we get that $\mathcal{H}_+ \subseteq \mathcal{D}[A_K]$. Now (4.7) yields that A_F and A_K are transversal. □

Corollary 4.7. *If a non-negative densely-defined symmetric operator $\dot{A}$ admits a non-negative self-adjoint bi-extension, then it also admits a non-negative self-adjoint bi-extension $\mathbb{A}$ with quasi-kernel A_K.*

It follows from Theorem 4.6 that if $A_K = A_F$, then the operator $\dot{A}$ does not admit non-negative self-adjoint bi-extensions. Consequently, in this case the analogue of the Friedrichs theorem is not true. The following theorem provides a criterion on when the analogue of the Friedrichs theorem does take place.

Theorem 4.8. *A non-negative densely-defined symmetric operator $\dot{A}$ admits a non-negative self-adjoint bi-extensions if and only if*

$$\int_0^\infty t\, d(E(t)h, h) < \infty \quad \text{for all} \quad h \in \mathfrak{N}_{-a},\ a > 0, \tag{4.8}$$

where $E(t)$ is a spectral function of the Kreĭn-von Neumann extension A_K of $\dot{A}$.

Proof. The inequality (4.8) is equivalent to the inclusion

$$\mathfrak{N}_{-a} \subset \mathrm{Dom}(A_K^{1/2}) = \mathcal{D}[A_K].$$

Since $-a$ is a regular point of A_K, the direct decomposition

$$\mathrm{Dom}(\dot{A}^*) = \mathrm{Dom}(A_K) \dot{+} \mathfrak{N}_{-a},$$

holds. So, from (4.7) we get that (4.8) is equivalent to transversality of A_F and A_K. The latter is equivalent to existence of non-negative self-adjoint bi-extension of $\dot{A}$ (see Theorem 4.6). □

Observe, that since $\mathfrak{N}_{-a}$ is a subspace in $\mathcal{H}$, A_K is closed in $\mathcal{H}$, condition (4.8) is equivalent to the following: there exists a positive number $k > 0$, depending on a, such that

$$\int_0^\infty t\, d(E(t)h, h) < k||h||^2, \quad \forall h \in \mathfrak{N}_{-a},\ a > 0. \tag{4.9}$$

On the other hand, (4.8) is equivalent (see proof of Theorem 4.6) to the existence of $k > 0$ such that

$$\int_0^\infty t\, d(E(t)f, f) < k||f||_+^2, \quad \forall f \in \mathrm{Dom}(\dot{A}^*).$$

5. Accretive bi-extensions

Let $\dot{A}$ be a densely defined and closed non-negative symmetric operator. In this section we will study the existence of accretive $(*)$-extensions of a given maximal accretive operator $T \in \Lambda(\dot{A})$.

Theorem 5.1. *If $\mathbb{A}$ is a quasi-self-adjoint bi-extension of $T \in \Omega(\dot{A})$ generated by $\tilde{A}$ via* (2.1)*, then for all $\phi = h + f \in \mathcal{H}_+$, $h \in \mathrm{Dom}(T)$, and $f \in \mathrm{Dom}(\tilde{A})$ we have*

$$(\mathbb{A}\phi, \phi) = (Th, h) + (\tilde{A}f, f) + 2\mathrm{Re}\,(Th, f). \tag{5.1}$$

Proof. Let $\mathbb{A} = \dot{A}^* - \mathcal{R}^{-1}\dot{A}^*(I - \mathcal{P}_{T\tilde{A}})$ according to (2.1). Note that for any $f \in \mathrm{Dom}(\tilde{A})$ we have that $P_{T\tilde{A}}f \in \mathrm{Dom}(\dot{A})$ and hence

$$(\mathcal{P}_{T\tilde{A}}f, Th) = (\dot{A}\mathcal{P}_{T\tilde{A}}f, h),$$

for any $h \in \mathrm{Dom}(T)$. Besides,

$$\left(\mathcal{R}^{-1}\dot{A}^*(I - \mathcal{P}_{T\tilde{A}})f, g\right) = 0, \qquad \forall f, g \in \mathrm{Dom}(\tilde{A}).$$

Indeed, since $\mathrm{Dom}(\tilde{A}) = \mathrm{Dom}(\dot{A}) \oplus (U + I)\mathfrak{N}_i$, where U is a unitary operator from $\mathfrak{N}_i$ onto $\mathfrak{N}_{-i}$, we have $(I - \mathcal{P}_{T\tilde{A}})f = (I + U)\varphi$, for $\varphi \in \mathfrak{N}_i$, and

$$\dot{A}^*(I - \mathcal{P}_{T\tilde{A}})f = i(I - U)\varphi.$$

Moreover, from the $(+)$-orthogonality of $(U + I)\mathfrak{N}_i$ and $(U - I)\mathfrak{N}_i$ we obtain the desired equation. Further,

$$\begin{aligned}(\mathbb{A}\phi, \phi) &= (Th + \tilde{A}f - \mathcal{R}^{-1}\tilde{A}(I - \mathcal{P}_{T\tilde{A}})f, h + f)\\ &= (Th, h) + (\tilde{A}f, f) + (Th, f) - (\tilde{A}(I - \mathcal{P}_{T\tilde{A}})f, h)_+ + (\tilde{A}f, h),\end{aligned}$$

and

$$\begin{aligned}(\tilde{A}(I - \mathcal{P}_{T\tilde{A}})f, h)_+ &= (\tilde{A}(I - \mathcal{P}_{T\tilde{A}})f, h) - ((I - \mathcal{P}_{T\tilde{A}})f, Th)\\ &= (\tilde{A}f, h) - (\dot{A}\mathcal{P}_{T\tilde{A}}f, h) - (f, Th) + (\mathcal{P}_{T\tilde{A}}f, Th)\\ &= (\tilde{A}f, h) - (f, Th).\end{aligned}$$

Consequently, $(\mathbb{A}\phi, \phi) = (Th, h) + (\tilde{A}f, f) + 2\mathrm{Re}\,(Th, f)$. □

Corollary 5.2. *Let T be a quasi-self-adjoint maximal accretive extension of $\dot{A}$. Assume that $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ is given by* (2.1) *and generated by a self-adjoint extension A transversal to T. Then $\mathbb{A}$ is accretive if and only if the form*

$$\mathrm{Re}\,(Th, h) + (Ag, g) + 2\,\mathrm{Re}\,(Th, g), \tag{5.2}$$

is non-negative for all $h \in \mathrm{Dom}(T)$ and $g \in \mathrm{Dom}(A)$.

By $\Xi(\dot{A})$ we denote the set of all maximal accretive quasi-self-adjoint extensions of the operator $\dot{A}$. In particular, the class $\Xi(\dot{A})$ contains all nonnegative self-adjoint extensions of $\dot{A}$. It follows from Lemma 5.2 that if $T \in \Xi(\dot{A})$ and if $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_+]$ of the form (2.1) is accretive, then $A \in \Xi(\dot{A})$. On the class $\Xi(\dot{A})$ we define Cayley transform given by the formula

$$K(T) = (I - T)(I + T)^{-1}, \qquad T \in \Xi(\dot{A}). \tag{5.3}$$

This Cayley transform sets one-to-one correspondence between the class $\Xi(\dot{A})$ and the set of quasi-self-adjoint contractive (qsc) extensions of a symmetric contraction

$$\dot{S} = (I - \dot{A})(I + \dot{A})^{-1},$$

defined on a subspace $\mathrm{Dom}(\dot{S}) = (I + \dot{A})\mathrm{Dom}(\dot{A})$, i.e., both Q and Q^* are extensions of S and $||Q|| \le 1$. Put

$$\mathfrak{N} = \mathcal{H} \ominus \mathrm{Dom}(\dot{S}). \tag{5.4}$$

Notice that $\mathfrak{N} = \mathfrak{N}_{-1} = \ker(\dot{A}^* + I)$ (the deficiency subspace of $\dot{A}$).

Let $S_\mu = K(A_F)$ and $S_M = K(A_K)$. It was shown in [7], [8], [10] that $Q \in [\mathcal{H}, \mathcal{H}]$ is a qsc-extension of a symmetric contraction $\dot{S}$ if and only if it can be represented in the form

$$Q = \frac{1}{2}(S_M + S_\mu) + \frac{1}{2}(S_M - S_\mu)^{1/2} X (S_M - S_\mu)^{1/2}, \tag{5.5}$$

where X is a contraction in the subspace $\overline{\mathrm{Ran}(S_M - S_\mu)} \subseteq \mathfrak{N}$.

Clearly, if X is a self-adjoint contraction, then (5.5) provides a description of all sc-extensions of a symmetric contraction S.

Lemma 5.3.

1) *The class $\Xi(\dot{A})$ contains mutually transversal operators if and only if A_F and A_K are mutually transversal.*
2) *Let T_1 and T_2 belong to $\Xi(\dot{A})$. Then T_1 and T_2 are mutually transversal if and only if*

$$(K(T_1) - K(T_2))\mathfrak{N} = \mathfrak{N}.$$

Proof. It follows from (5.5) that

$$K(T_1) - K(T_2) = \frac{1}{2}(S_M - S_\mu)^{1/2}(X_1 - X_2)(S_M - S_\mu)^{1/2},$$

where X_l, $(l = 1, 2)$ are the corresponding to T_l contractions in $\overline{\mathrm{Ran}(S_M - S_\mu)}$. Relation (5.3) yields

$$K(T_1) - K(T_2) = 2\left((I + T_1)^{-1} - (I + T_2)^{-1}\right).$$

Thus

$$(I + T_1)^{-1} - (I + T_2)^{-1} = \frac{1}{4}(S_M - S_\mu)^{1/2}(X_1 - X_2)(S_M - S_\mu)^{1/2}.$$

Furthermore, using [35] we get that

$$\begin{aligned}&\left((I+T_1)^{-1}-(I+T_2)^{-1}\right)\mathfrak{N}_{-1}=\mathfrak{N}_{-1}\\ \iff &\begin{cases}\overline{\mathrm{Ran}(S_M-S_\mu)}=\mathrm{Ran}(S_M-S_\mu)=\mathfrak{N}=\mathfrak{N}_{-1},\\ \qquad\qquad\qquad\qquad\mathrm{Ran}(X_1-X_2)\mathfrak{N}=\mathfrak{N}.\end{cases}\end{aligned}\qquad\square$$

In what follows we assume that A_K and A_F are mutually transversal. Let A_1 and A_2 be two mutually transversal operators from $\Xi(\dot{A})$. Consider a form defined on $\mathrm{Dom}(A_1)\times\mathrm{Dom}(A_2)$ as follows

$$B(f_1,f_2)=(A_1f_1,f_1)+(A_2f_2,f_2)+2\mathrm{Re}\,(A_1f_1,f_2),\tag{5.6}$$

where $f_l\in\mathrm{Dom}(A_l)$, $(l=1,2)$. Let

$$\phi_l=\frac{1}{2}(I+A_l)f_l,\qquad S_l\phi_l=\frac{1}{2}(I-A_l)f_l,$$

be the Cayley transform of A_l for $l=1,2$. Then

$$f_l=(I+S_l)\phi_l,\qquad A_lf_l=(I-S_l)\phi_l,\quad (l=1,2).\tag{5.7}$$

Substituting (5.7) into (5.6) we obtain a form defined on $\mathcal{H}\times\mathcal{H}$

$$\tilde{B}(\phi_1,\phi_2)=\|\phi_1+\phi_2\|^2-\|S_1\phi_1+S_2\phi_2\|^2-2\mathrm{Re}\,\left((S_1-S_2)\phi_1,\phi_2\right).$$

Let us set

$$F=\frac{1}{2}(S_1-S_2),\quad G=\frac{1}{2}(S_1+S_2),\quad u=\frac{1}{2}(\phi_1+\phi_2),\quad v=\frac{1}{2}(\phi_1-\phi_2).\tag{5.8}$$

Then $\tilde{B}(\phi_1,\phi_2)=4H(u,v)$ where

$$H(u,v)=\|u\|^2+(Fv,v)-(Fu,u)-\|Fv+Gu\|^2.\tag{5.9}$$

Moreover, $F\pm G$ are contractive operators. From the above reasoning we conclude that non-negativity of the form $B(f_1,f_2)$ on $\mathrm{Dom}(A_1)\times\mathrm{Dom}(A_2)$ is equivalent to non-negativity of the form $H(u,v)$ on $\mathcal{H}\times\mathcal{H}$.

Lemma 5.4. *The form $H(u,v)$ in* (5.9) *is non-negative for all $u,v\in\mathcal{H}$ if and only if operator F defined in* (5.8) *is non-negative.*

Proof. If $H(u,v)\geq 0$ for all $u,v\in\mathcal{H}$ then $H(0,v)\geq 0$ for all $v\in\mathcal{H}$. Hence $(Fv,v)\geq\|Fv\|^2\geq 0$, i.e., $F\geq 0$.

Conversely, let $F\geq 0$. Since both operators $F\pm G$ are self-adjoint contractions, then $-I\leq F+G\leq I$ and $-I\leq F-G\leq I$. This implies $-(I-F)\leq G\leq I-F$, and thus

$$G=(I-F)^{1/2}X(I-F)^{1/2},\tag{5.10}$$

where X is a self-adjoint contraction. Then (5.10) yields that for all $u,v\in\mathcal{H}$

$$\begin{aligned}\|Fv+Gu\|&=\|Fv\|^2+\|Gu\|^2+2\mathrm{Re}\,(Fv,Gu)\\ &\leq\|Fv\|^2+\left((I-F)X(I-F)^{1/2}u,X(I-F)^{1/2}u\right)\\ &\quad+2\left|\left(F(I-F)^{1/2}X(I-F)^{1/2}u,v\right)\right|\end{aligned}$$

$$
\begin{aligned}
&= \|Fv\|^2 + \|X(I-F)^{1/2}u\|^2 - (FX(I-F)^{1/2}u, X(I-F)^{1/2}u)\\
&\quad + 2\left|\Big(FX(I-F)^{1/2}u, (I-F)^{1/2}v\Big)\right|\\
&\le \|Fv\|^2 + \|X(I-F)^{1/2}u\|^2 - (FX(I-F)^{1/2}u, X(I-F)^{1/2}u)\\
&+ (FX(I-F)^{1/2}u, X(I-F)^{1/2}u) + (F(I-F)^{1/2}v, (I-F)^{1/2}v)\\
&= \|Fv\|^2 + \|X(I-F)^{1/2}u\|^2 + (Fv,v) - \|Fv\|^2\\
&\le (Fv,v) + \|u\|^2 - (Fu,u).
\end{aligned}
$$

Therefore, for all $u, v \in \mathcal{H}$

$$
H(u,v) = \|u\|^2 - (Fu,u) + (Fv,v) - \|Fv + Gu\|^2 \ge 0.
$$

The lemma is proved. $\square$

Theorem 5.5. *Let $\mathbb{A} = \dot{A}^* - \mathcal{R}^{-1}\dot{A}^*(I - \mathcal{P}_{\hat{A}A})$ be a self-adjoint $(*)$-extension of a non-negative symmetric operator $\dot{A}$, with a self-adjoint quasi-kernel $\hat{A} \in \Xi(\dot{A})$, and generated (via (2.1)) by a self-adjoint extension A. Then the following statements are equivalent*

(i) $\mathbb{A}$ *is non-negative*
(ii) $\Big(K(\hat{A}) - K(A)\Big) \upharpoonright \mathfrak{N}$, *(where $\mathfrak{N}$ is defined by (5.4)) is positively defined,*
(iii) $(\hat{A} + I)^{-1} \ge (A + I)^{-1}$, *and $\hat{A}$ is transversal to A,*
(iv) $\hat{A} \le A$ *and $\hat{A}$ is transversal to A.*

Proof. Let $(\mathbb{A}f, f) \ge 0$ for all $f \in \mathcal{H}_+$. Then due to (5.1) and Corollary 5.2 we have that the form

$$
B(g,h) = (\hat{A}g, h) + (Ah, g) + 2\mathrm{Re}\,(\hat{A}g, h), \quad (g \in \mathrm{Dom}(\hat{A}),\ h \in \mathrm{Dom}(A)),
$$

is non-negative on $\mathrm{Dom}(\hat{A}) \times \mathrm{Dom}(A)$. Consequently, the form $H(u,v)$ given by (5.9) is non-negative for all $u, v \in \mathcal{H}$ where

$$
F = \frac{1}{2}\Big(K(\hat{A}) - K(A)\Big) \text{ and } G = \frac{1}{2}\Big(K(\hat{A}) + K(A)\Big).
$$

Using Lemma 5.4 we conclude that $F\upharpoonright\mathfrak{N} \ge 0$ and applying Lemma 5.3 yields $F\mathfrak{N} = \mathfrak{N}$. This proves that (i) $\Rightarrow$ (ii). The implication (ii) $\Rightarrow$ (i) can be shown by reversing the argument. Since

$$
K(\hat{A}) = -I + 2(\hat{A} + I)^{-1},\ K(A) = -I + 2(A + I)^{-1},
$$

we get that (ii) $\Longleftrightarrow$ (iii). Applying inequalities from [24] yields that (iii) $\Longleftrightarrow$ (iv). $\square$

Theorem 5.6. *A self-adjoint operator $\hat{A} \in \Xi(\dot{A})$ admits non-negative $(*)$-extensions if and only if $\hat{A}$ is transversal to A_F.*

Proof. If $\hat{A}$ is transversal to A_F, then $\Big(K(\hat{A}) - K(A_F)\Big)\upharpoonright\mathfrak{N}$ is positively defined. Applying Theorem 5.5 we obtain that

$$\mathbb{A} = \dot{A}^* - \mathcal{R}^{-1}\dot{A}^*(I - \mathcal{P}_{\hat{A}A_F}),$$

is a non-negative (∗)-extension.

Conversely, if $\mathbb{A} = \dot{A}^* - \mathcal{R}^{-1}\dot{A}^*(I - \mathcal{P}_{\hat{A}A})$ is a (∗)-extension of $\hat{A}$, then via Theorem 5.5 we get that $\Big(K(\hat{A}) - K(A)\Big)\upharpoonright\mathfrak{N}$ is positively defined. But then due to the chain of inequalities

$$K(\hat{A}) \geq K(A) \geq K(A_F),$$

the operator $\Big(K(\hat{A}) - K(A_F)\Big)\upharpoonright\mathfrak{N}$ is positively defined as well. According to Lemma 5.3 $\hat{A}$ is transversal A_F. □

We note that if $\hat{A}$ is a self-adjoint extension of $\dot{A}$, then all self-adjoint (∗)-extensions of $\hat{A}$ coincide with t-self-adjoint bi-extensions of $\dot{A}$ with the quasi-kernel $\hat{A}$. Consequently, Theorem 5.6 gives the criterion of the existence of a non-negative t-self-adjoint bi-extension of $\dot{A}$ and hence provides the conditions when Friedrichs theorem for t-self-adjoint bi-extensions is true.

Now we focus on non-self-adjoint accretive (∗)-extensions of operator $T \in \Xi(\dot{A})$.

Lemma 5.7. *Let $\mathbb{A}$ be a (∗)-extensions of operator $T \in \Xi(\dot{A})$ generated by an operator $A \in \Xi(\dot{A})$. Then the quasi-kernel $\hat{A}$ of the operator $\operatorname{Re}\mathbb{A}$ is defined by the formula*

$$\begin{aligned} f &= (Q+I)g + \frac{1}{2}(S+I)(Q^* - S)^{-1}(Q - Q^*)g, \\ \hat{A}f &= (I-Q)g + \frac{1}{2}(I-S)(Q^* - S)^{-1}(Q - Q^*)g, \end{aligned} \tag{5.11}$$

where $g \in \mathcal{H}$, $Q = K(T)$, $Q^* = K(T^*)$, and $S = K(A)$.

Proof. Let $\mathbb{A} = \dot{A}^* - \mathcal{R}^{-1}\dot{A}^*(I - \mathcal{P}_{TA})$ (of the form (2.1)) be a (∗)-extensions of operator T generated by a self-adjoint extension A. Let

$$\operatorname{Dom}(A) = \operatorname{Dom}(\dot{A}) \oplus (\mathcal{U} + I)\mathfrak{N}_i,$$

where $\mathcal{U} \in [\mathfrak{N}_i, \mathfrak{N}_{-i}]$ is a unitary mapping. Suppose $f \in \operatorname{Dom}(\hat{A})$, where $\hat{A}$ is a quasi-kernel of $\operatorname{Re}\mathbb{A}$. Due to the transversality of T^* and A and T and A we have

$$f = u + (\mathcal{U} + I)\varphi, \qquad f = v + (\mathcal{U} + I)\psi,$$

where $u \in \operatorname{Dom}(T)$, $v \in \operatorname{Dom}(T^*)$, and $\varphi, \psi \in \mathfrak{N}_i$. Also

$$\begin{aligned} \mathbb{A}f &= Tu + \dot{A}^*(\mathcal{U} + I)\varphi - i\mathcal{R}^{-1}(I - \mathcal{U})\varphi, \\ \mathbb{A}^*f &= T^*v + \dot{A}^*(\mathcal{U} + I)\psi - i\mathcal{R}^{-1}(I - \mathcal{U})\psi, \end{aligned}$$

and

$$\begin{aligned}\hat{A}f &= \frac{1}{2}(\mathbb{A}f + \mathbb{A}^* f)\\ &= \frac{1}{2}\left(Tu + T^*v + \dot{A}^*(\mathcal{U}+I)\varphi + \dot{A}^*(\mathcal{U}+I)\psi - i\mathcal{R}^{-1}(I-\mathcal{U})(\varphi+\psi)\right).\end{aligned}$$

Since $\hat{A}f \in \mathcal{H}$, then $\varphi = -\psi$ and hence any vector $f \in \mathrm{Dom}(\hat{A})$ is uniquely represented in the form

$$f = u + \phi, \quad u \in \mathrm{Dom}(T),\ \phi \in (\mathcal{U}+I)\mathfrak{N}_i,$$

or in the form $f = v - \phi$, $v \in \mathrm{Dom}(T^*)$. By (2.1) (see also [11]) $\hat{A}$ is transversal to A and

$$\mathrm{Re}\,\mathbb{A} = \dot{A}^* - \mathcal{R}^{-1}(I - P_{\hat{A}A}).$$

Thus $\mathfrak{M}_{\hat{A}} \dotplus (\mathcal{U}+I)\mathfrak{N}_i = \mathfrak{M}$, where $\mathfrak{M}_{\hat{A}} = \mathrm{Dom}(\hat{A}) \ominus \mathrm{Dom}(\dot{A})$. It follows from

$$\mathfrak{M}_T \dotplus (\mathcal{U}+I)\mathfrak{N}_i = \mathfrak{M}, \quad \text{where} \quad \mathfrak{M}_T = \mathrm{Dom}(T) \ominus \mathrm{Dom}(\dot{A}),$$

that $\mathcal{P}_{\hat{A}A}\mathfrak{M}_T = \mathfrak{M}_{\hat{A}}$ and hence for any $u \in \mathrm{Dom}(T)$ there exists a $\phi \in (\mathcal{U}+I)\mathfrak{N}_i$ and $f \in \mathrm{Dom}(\hat{A})$ such that $f = u + \phi$. Similarly, for any $v \in \mathrm{Dom}(T)$ there exists a $\phi \in (\mathcal{U}+I)\mathfrak{N}_i$ and $f \in \mathrm{Dom}(\hat{A})$ such that $f = u - \phi$. Since

$$\mathrm{Dom}(T) = (I+Q)\mathcal{H},\ \mathrm{Dom}(T^*) = (I+Q^*)\mathcal{H},\ \mathrm{Dom}(A) = (I+S)\mathcal{H},$$

and

$$Q\restriction\mathrm{Dom}(\dot{S}) = Q^*\restriction\mathrm{Dom}(\dot{S}) = S\restriction\mathrm{Dom}(\dot{S}),$$

we conclude that for any $f \in \mathrm{Dom}(\hat{A})$ there are uniquely defined $g, g_* \in \mathcal{H}$ and $h \in \mathfrak{N}$ such that

$$f = (Q+I)g + (S+I)h, \quad f = (Q^*+I)g_* - (S+I)h. \tag{5.12}$$

Conversely, for every $g \in \mathcal{H}$ (respectively, $g_* \in \mathcal{H}$) there are $g_* \in \mathcal{H}$ (respectively, $g \in \mathcal{H}$) and $h \in \mathfrak{N}$, such that (5.12) holds with $f \in \mathrm{Dom}(\hat{A})$. Since $\dot{A}^*(Q+I)g = (I-Q)g$, $\dot{A}^*(I+Q^*)g_* = (I-Q^*)g_*$, and $\dot{A}^*(I+S)h = (I-S)h$, then

$$\hat{A}f = (I-Q)g + (I-S)h, \quad \hat{A}f = (I-Q^*)g_* - (I-S)h. \tag{5.13}$$

From (5.12) and (5.13) we have $2h = g_* - g$ and $2Sh = Q^*g_* - g$, which implies

$$2(Q^*-S)h = (Q-Q^*)g. \tag{5.14}$$

Since T^* and A are mutually transversal, according to Lemma 5.3 $(Q^*-S)\restriction\mathfrak{N}$ is an isomorphism on $\mathfrak{N}$. Then (5.14) implies

$$h = \frac{1}{2}(Q^*-S)^{-1}(Q-Q^*)g. \tag{5.15}$$

Substituting (5.15) into (5.12) and (5.13) we obtain (5.11). □

Lemma 5.8. *Let $T \in \Xi(\dot A)$ and $A \in \Xi(\dot A)$ be a transversal to T self-adjoint operator. If the operator*

$$[K(T) + K(T^*) - 2K(A)]\restriction \mathfrak{N},$$

is an isomorphism of the space $\mathfrak{N}$ (defined in (5.4)), then the quasi-kernel $\hat A$ of the real part of the operator $\mathbb{A} = \dot A^* - \mathcal{R}^{-1}\dot A^*(I - \mathcal{P}_{TA})$ is a Cayley transform of the operator

$$\hat S = S + (Q - S)(\operatorname{Re} Q - S)^{-1}(Q^* - S),$$

where $S = K(A)$ and $\operatorname{Re} Q = (1/2)[K(T) + K(T^*)]$.

Proof. Let $\mathbb{A} = \dot A^* - \mathcal{R}^{-1}\dot A^*(I - \mathcal{P}_{TA})$. Then by the virtue of Lemma 5.7, formula (5.11) defines the quasi-kernel $\hat A$ of the operator $\operatorname{Re}\mathbb{A}$. It also follows from (5.11) that

$$f + \hat A f = 2g + (Q^* - S)^{-1}(Q - Q^*)g.$$

Let $P_{\mathfrak{N}}$ and $P_{\dot S}$ denote the orthoprojection operators in $\mathcal{H}$ according to (5.4) onto $\mathfrak{N}$ and $\operatorname{Dom}(\dot S)$, respectively. Then

$$\begin{aligned} 2g + (Q^* - S)^{-1}(Q - Q^*)g &= 2P_{\dot S}g + (Q^* - S)^{-1}(2Q^* - 2S + Q - Q^*)P_{\mathfrak{N}}g \\ &= 2P_{\dot S}g + 2(Q^* - S)^{-1}(\operatorname{Re} Q - S)P_{\mathfrak{N}}g, \end{aligned}$$

and

$$(I + \hat A)f = 2P_{\dot S}g + 2(Q^* - S)^{-1}(\operatorname{Re} Q - S)P_{\mathfrak{N}}g. \tag{5.16}$$

From the statement of the lemma we have that $(\operatorname{Re} Q - S)\restriction \mathfrak{N}$ is an isomorphism of the space $\mathfrak{N}$. Hence, (5.16) $\operatorname{Ran}(I + \hat A) = \mathcal{H}$ and the Cayley transform is well defined for $\hat A$. Let

$$\hat S = (I - \hat A)(I + \hat A)^{-1}.$$

It follows from (5.11) that

$$\begin{aligned} (\hat S + I)\phi &= (Q + I)g + \tfrac{1}{2}(S + I)(Q^* - S)^{-1}(Q - Q^*)g, \\ (I - \hat S)\phi &= (I - Q)g + \tfrac{1}{2}(I - S)(Q^* - S)^{-1}(Q - Q^*)g. \end{aligned}$$

Therefore,

$$\begin{aligned} \phi &= g + \tfrac{1}{2}(Q^* - S)^{-1}(Q - Q^*)g, \\ \hat S\phi &= Qg + \tfrac{1}{2}S(Q^* - S)^{-1}(Q - Q^*)g. \end{aligned}$$

and hence

$$\begin{aligned} \hat S\phi &= S\phi + (Q - S)P_{\mathfrak{N}}g, \\ \phi &= P_{\dot S}g + (Q^* - S)^{-1}(\operatorname{Re} Q - S)P_{\mathfrak{N}}g. \end{aligned} \tag{5.17}$$

Using the second half of (5.17) we have

$$P_{\mathfrak{N}}g = (\operatorname{Re} Q - S)^{-1}(Q^* - S)P_{\mathfrak{N}}\phi. \tag{5.18}$$

Substituting, (5.18) into the first part of (5.17) we obtain

$$\hat S\phi = S\phi + (Q - S)(\operatorname{Re} Q - S)^{-1}(Q^* - S)\phi,$$

which proves the lemma. □

Let $T \in \Xi(\dot{A})$. By the **class** Ξ_{AT} we denote the set of all non-negative self-adjoint operators $A \supset \dot{A}$ satisfying the following conditions:

1. $[K(T) + K(T^*) - 2K(A)]\upharpoonright \mathfrak{N}$ is a non-negative operator in $\mathfrak{N}$, where $\mathfrak{N}$ is defined in (5.4);
2. $K(A)+2[K(T)-K(A)][K(T)+K(T^*)-2K(A)]^{-1}[K(T^*)-K(A)] \le K(A_K)$.[1]

Theorem 5.9. *A $(*)$-extension of operator $T \in \Xi(\dot{A})$*

$$\mathbb{A} = \dot{A}^* - \mathcal{R}^{-1}\dot{A}^*(I - \mathcal{P}_{TA}),$$

generated by a self-adjoint operator $A \supset \dot{A}$ is accretive if and only if $A \in \Xi_{AT}$.

Proof. We prove the necessity part first. Let $\mathbb{A} = \dot{A}^* - \mathcal{R}^{-1}\dot{A}^*(I - \mathcal{P}_{TA})$ be an accretive $(*)$-extension, then $\operatorname{Re}\mathbb{A}$ is a non-negative $(*)$-extension of the quasi-kernel $\hat{A} \in \Xi(\dot{A})$. But according to Lemma 5.7 $\hat{A}$ is defined by formulas (5.11). Since (-1) is a regular point of operator $\hat{A}$, then (5.16) implies that the operator

$$(\operatorname{Re} Q - S)\upharpoonright \mathfrak{N} = \frac{1}{2}[K(T) + K(T^*) - 2K(A)]\upharpoonright \mathfrak{N},$$

is an isomorphism of the space $\mathfrak{N}$. According to Lemma 5.8 we have

$$K(\hat{A}) = K(A) + 2[K(T) - K(A)][K(T) + K(T^*) - 2K(A)]^{-1}[K(T^*) - K(A)].$$

Since $K(\hat{A})$ is a self-adjoint contractive extension of $\dot{S}$, then $K(\hat{A}) \le K(A_K)$. Also, since $\operatorname{Re}\mathbb{A}$ is generated by A and $\operatorname{Re}\mathbb{A} \ge 0$, then by Theorem 5.5 the operator $[K(\hat{A}) - K(A)]\upharpoonright \mathfrak{N}$ is non-negative. Consequently, the operator $[K(T) + K(T^*) - 2K(A)\upharpoonright \mathfrak{N}$ is non-negative as well and we conclude that $A \in \Xi_{AT}$.

Now we prove sufficiency. Let $A \in \Xi_{AT}$, then by Lemma 5.8, $\hat{A}$ is a Cayley transform of a self-adjoint extension $\hat{S}$ of the operator $\dot{S}$. Since

$$[\hat{S} - S]\upharpoonright \mathfrak{N} = [K(\hat{A}) - K(A)]\upharpoonright \mathfrak{N},$$

is a non-negative operator, then due to Theorem 5.5 the operator $\operatorname{Re}\mathbb{A}$ is a non-negative $(*)$-extension of $\hat{A}$. That is why $\mathbb{A}$ is an accretive $(*)$-extension of operator T. □

Theorem 5.10. *An operator $T \in \Xi(\dot{A})$ admits accretive $(*)$-extensions if and only if T is transversal to A_F.*

Proof. If T admits accretive $(*)$-extensions, then the class Ξ_{AT} is non-empty, i.e., there exists a self-adjoint operator $A \in \Xi(\dot{A})$ such that the operator

$$[(K(T) + K(T^*) - 2K(A)]\upharpoonright \mathfrak{N},$$

is non-negative. But then $[(K(T) + K(T^*) - 2K(A_F)]\upharpoonright \mathfrak{N}$ is non-negative as well. This yields that $[K(T) - K(A_F)]\upharpoonright \mathfrak{N}$ is an isomorphism of $\mathfrak{N}$. Then by Lemma 5.3 T and A_F are mutually transversal. This proves the necessity.

[1] When we write $[K(T) + K(T^*) - 2K(A)]^{-1}$ we mean the operator inverse to $[K(T) + K(T^*) - 2K(A)]\upharpoonright \mathfrak{N}$.

Now let us assume that T and A_F are mutually transversal. We will show that in this case $A_F \in \Xi_{AT}$. By Lemma 5.3, $[K(T)-K(A_F)]\upharpoonright\mathfrak{N}$ is an isomorphism of the space $\mathfrak{N}$. Then using formula (5.5) we have

$$K(T) = Q = \frac{1}{2}(S_M + S_\mu) + \frac{1}{2}(S_M - S_\mu)^{1/2} X (S_M - S_\mu)^{1/2},$$

where $X \in [\mathfrak{N}, \mathfrak{N}]$ is a contraction. Furthermore,

$$(Q - S_\mu)\upharpoonright\mathfrak{N} = \frac{1}{2}(S_M - S_\mu)^{1/2}(X + I)(S_M - S_\mu)^{1/2}\upharpoonright\mathfrak{N}.$$

Thus, $X + I$ is an isomorphism of the space $\mathfrak{N}$. Moreover, $\operatorname{Re} X + I \geq 0$ and for every $f \in \mathfrak{N}$

$$((\operatorname{Re} X + I)f, f) = \frac{1}{2}\left(\|f\|^2 - \|Xf\|^2 + \|(X + I)f\|^2\right). \tag{5.19}$$

But since $\|(X + I)f\|^2 \geq a\|f\|^2$, where $a > 0$, $f \in \mathfrak{N}$, we have

$$((\operatorname{Re} X + I)f, f) \geq b\|f\|^2, \qquad (b > 0).$$

Hence, $\operatorname{Re} X + I$ is a non-negative operator implying that

$$[(1/2)(K(T) + K(T^*) - K(A)]\upharpoonright\mathfrak{N} = \frac{1}{2}(S_M - S_\mu)^{1/2}(\operatorname{Re} X + I)(S_M - S_\mu)^{1/2}\upharpoonright\mathfrak{N},$$

is non-negative too. Also (5.19) implies

$$\operatorname{Re} X + I \geq \frac{1}{2}(X^* + I)(X + I).$$

It is easy to see then that $(\operatorname{Re} X + I)^{-1} \leq 2(X + I)^{-1}(X^* + I)^{-1}$. Therefore,

$$\frac{1}{2}(X + I)(\operatorname{Re} X + I)^{-1}(X^* + I) \leq I. \tag{5.20}$$

Now, since

$$\begin{aligned} K(A_F) &+ 2[K(T) - K(A)][K(T) + K(T^*) - 2K(A)]^{-1}[K(T^*) - K(A)] \\ &= S_\mu + \frac{1}{2}(S_M - S_\mu)^{1/2}(X + I)(\operatorname{Re} X + I)^{-1}(X^* + I)(S_M - S_\mu)^{1/2}, \end{aligned}$$

then applying (5.20) we obtain

$$K(A_F) + 2[K(T) - K(A)][K(T) + K(T^*) - 2K(A)]^{-1}[K(T^*) - K(A)] \leq K(A_K).$$

Thus, A_F belongs to the class Ξ_{AT} and applying theorem (5.9) we conclude that $\mathbb{A} = \dot{A}^* - \mathcal{R}^{-1}\dot{A}^*(I - \mathcal{P}_{AT})$ is an accretive (∗)-extension of T. □

A qsc-extension

$$Q = \frac{1}{2}(S_M + S_\mu) + \frac{1}{2}(S_M - S_\mu)^{1/2} X (S_M - S_\mu)^{1/2}$$

is called **extremal** if X is isometry.

Theorem 5.11. *Let $T \in \Xi(\dot{A})$ be transversal to A_F. Then the accretive (∗)-extension $\mathbb{A}$ of T generated by A_F has a property that $\operatorname{Re}\mathbb{A} \supset A_K$ if and only if T and T^* are extremal extensions of $\dot{A}$.*

Proof. Suppose $\mathrm{Re}\,\mathbb{A} \supset A_K$ and $\mathrm{Re}\,\mathbb{A} = \dot{A}^* - \mathcal{R}^{-1}\dot{A}^*(I - \mathcal{P}_{TA_F})$. Then by Lemma 5.8 we have

$$S_M = S_\mu + (Q - S_\mu)(\mathrm{Re}\,Q - S_\mu)^{-1}(Q^* - S_\mu).$$

Thus,

$$(X + I)(\mathrm{Re}\,X + I)^{-1}(X^* + I) = 2I, \tag{5.21}$$

where

$$Q = K(T) = \frac{1}{2}(S_M + S_\mu) + \frac{1}{2}(S_M - S_\mu)^{1/2}X(S_M - S_\mu)^{1/2}.$$

It is easy to see that

$$(X^* + I)(\mathrm{Re}\,X + I)^{-1}(X + I) = (X + I)(\mathrm{Re}\,X + I)^{-1}(X^* + I). \tag{5.22}$$

Then it follows from (5.21) and (5.22) that

$$X^*X = XX^* = I,$$

i.e., X is a unitary operator in $\mathfrak{N}$. Then both operators T and T^* are extremal m-accretive extensions of $\dot{A}$.

The second part of the theorem is proved by reversing the argument. □

6. Realization of Stieltjes functions

Definition 6.1. An operator-valued Herglotz-Nevanlinna function $V(z)$ in a finite-dimensional Hilbert space E is called a **Stieltjes function** if $V(z)$ is holomorphic in $\mathrm{Ext}[0, +\infty)$ and

$$\frac{\mathrm{Im}[zV(z)]}{\mathrm{Im}\ z} \geq 0. \tag{6.1}$$

Consequently, an operator-valued Herglotz-Nevanlinna function $V(z)$ is Stieltjes if $zV(z)$ is also a Herglotz-Nevanlinna function. Applying the integral representation (1.1) (see also [17]) for this case we get that

$$\sum_{k,l=1}^{n} \left(\frac{z_k V(z_k) - \bar{z}_l V(\bar{z}_l)}{z_k - \bar{z}_l} h_k, h_l\right)_E \geq 0, \tag{6.2}$$

for an arbitrary sequence $\{z_k\}$ $(k = 1, \dots, n)$ of $(\mathrm{Im}\, z_k > 0)$ complex numbers and a sequence of vectors $\{h_k\}$ in E.

Similar to (1.1) formula holds true for the case of a Stieltjes function. Indeed, if $V(z)$ is a Stieltjes operator-valued function, then

$$V(z) = \gamma + \int_0^\infty \frac{dG(t)}{t - z}, \tag{6.3}$$

where $\gamma \geq 0$ and $G(t)$ is a non-decreasing on $[0, +\infty)$ operator-valued function such that

$$\int_0^\infty \frac{(dG(t)h, h)_E}{1 + t} < \infty, \quad h \in E. \tag{6.4}$$

Theorem 6.2. *Let Θ be an L-system of the form* (2.3) *with a densely defined non-negative symmetric operator $\dot{A}$. Then the impedance function $V_\Theta(z)$ defined by* (2.7) *is a Stieltjes function if and only if the operator $\mathbb{A}$ of the L-system Θ is accretive.*

Proof. Let us assume first that $\mathbb{A}$ is an accretive operator, i.e., $(\mathrm{Re}\,\mathbb{A}f, f) \geq 0$, for all $f \in \mathcal{H}_+$. Let $\{z_k\}$ $(k = 1, \dots, n)$ be a sequence of $(\mathrm{Im}\, z_k > 0)$ complex numbers and h_k be a sequence of vectors in E. Let us denote

$$Kh_k = \delta_k, \quad g_k = (\mathrm{Re}\,\mathbb{A} - z_k I)^{-1}\delta_k, \quad g = \sum_{k=1}^{n} g_k. \tag{6.5}$$

Since $(\mathrm{Re}\,\mathbb{A}g, g) \geq 0$, we have

$$\sum_{k,l=1}^{n} (\mathrm{Re}\,\mathbb{A}g_k, g_l) \geq 0. \tag{6.6}$$

By formal calculations one can have $(\mathrm{Re}\,\mathbb{A})g_k = \delta_k + z_k(\mathrm{Re}\,\mathbb{A} - z_k I)^{-1}\delta_k$, and

$$\begin{aligned}\sum_{k,l=1}^{n} (\mathrm{Re}\,\mathbb{A}g_k, g_l) = &\sum_{k,l=1}^{n} \big[(\delta_k, (\mathrm{Re}\,\mathbb{A} - z_l I)^{-1}\delta_l) \\ &+ (z_k(\mathrm{Re}\,\mathbb{A} - z_k I)^{-1}\delta_k, (\mathrm{Re}\,\mathbb{A} - z_k I)^{-1}\delta_l)\big].\end{aligned}$$

Using obvious equalities

$$\big((\mathrm{Re}\,\mathbb{A} - z_k I)^{-1} Kh_k, Kh_l\big) = \big(V_\Theta(z_k)h_k, h_l\big)_E,$$

and

$$\big((\mathrm{Re}\,\mathbb{A} - \bar{z}_l I)^{-1}(\mathrm{Re}\,\mathbb{A} - z_k I)^{-1} Kh_k, Kh_l\big) = \left(\frac{V_\Theta(z_k) - V_\Theta(\bar{z}_l)}{z_k - \bar{z}_l} h_k, h_l\right)_E,$$

we obtain

$$\sum_{k,l=1}^{n} ((\mathrm{Re}\,\mathbb{A})g_k, g_l) = \sum_{k,l=1}^{n} \left(\frac{z_k V_\Theta(z_k) - \bar{z}_l V_\Theta(\bar{z}_l)}{z_k - \bar{z}_l} h_k, h_l\right)_E \geq 0, \tag{6.7}$$

which implies that $V_\Theta(z)$ is a Stieltjes function.

Now we prove necessity. First we assume that $\dot{A}$ is a prime operator[2]. Then the equivalence of (6.7) and (6.6) implies that $(\mathrm{Re}\,\mathbb{A}g, g) \geq 0$ for any g from c.l.s.$\{\mathfrak{N}_z\}$. It was shown in [11] that a symmetric operator $\dot{A}$ with the equal deficiency indices is prime if and only if

$$\underset{z \neq \bar{z}}{c.l.s.}\, \mathfrak{N}_z = \mathcal{H}. \tag{6.8}$$

Thus $(\mathrm{Re}\,\mathbb{A}g, g) \geq 0$ for any $g \in \mathcal{H}_+$ and therefore $\mathbb{A}$ is an accretive operator.

Now let us assume that $\dot{A}$ is not a prime operator. Then there exists a subspace $\mathcal{H}^1 \subset \mathcal{H}$ on which $\dot{A}$ generates a self-adjoint operator A_1. Let us denote by

[2] We call a closed linear operator in a Hilbert space $\mathcal{H}$ a **prime operator** if there is no non-trivial reducing invariant subspace of $\mathcal{H}$ on which it induces a self-adjoint operator.

$\dot{A}_0$ an operator induced by $\dot{A}$ on $\mathcal{H}^0 = \mathcal{H} \ominus \mathcal{H}^1$. As it was shown shown in the proof of Theorem 12 of [14] the decomposition

$$\mathcal{H}_+ = \mathcal{H}^0_+ \oplus \mathcal{H}^1_+, \quad \mathcal{H}^0_+ = \mathrm{Dom}(\dot{A}^*_0),\ \mathcal{H}^1_+ = \mathrm{Dom}(A_1), \tag{6.9}$$

is (+)-orthogonal. Since $\dot{A}$ is a non-negative operator, then

$$(\mathrm{Re}\,\mathbb{A}g, g) = (A_1 g, g) = (\dot{A}g, g) \geq 0, \qquad \forall g \in \mathcal{H}^1_+ = \mathrm{Dom}(A_1).$$

On the other hand operator $\dot{A}_0$ is prime in $\mathcal{H}^0$ and hence $\underset{z \neq \bar{z}}{c.l.s.}\,\mathfrak{N}^0_z = \mathcal{H}^0$, where $\mathfrak{N}^0_z$ are the deficiency subspaces of the symmetric operator $\dot{A}_0$ in $\mathcal{H}^0$. Then the equivalence of (6.7) and (6.6) again implies that $(\mathrm{Re}\,\mathbb{A}g, g) \geq 0$ for any $g \in \mathcal{H}^0_+$. Taking into account decomposition (6.9) we conclude that $\mathrm{Re}\,(\mathbb{A}g, g) \geq 0$ holds for all $g \in \mathcal{H}_+$ and hence $\mathbb{A}$ is accretive. □

Now we define a class of realizable Stieltjes functions. At this point we need to note that since Stieltjes functions form a subset of Herglotz-Nevanlinna functions, then according to (1.7) and realization Theorems 8 and 9 of [14], we have that the class of all realizable Stieltjes functions is a subclass of $N(R)$. To see the specifications of this class we recall that aside of the integral representation (6.3), any Stieltjes function admits a representation (1.1). According to (1.7) a Herglotz-Nevanlinna operator-function can be realized if and only if in the representation (1.1) $L = 0$ and

$$Qh = \int_{-\infty}^{+\infty} \frac{t}{1+t^2}\, dG(t)h, \tag{6.10}$$

for all $h \in E$ such that

$$\int_{-\infty}^{\infty} (dG(t)h, h)_E < \infty. \tag{6.11}$$

holds. Considering this we obtain

$$Q = \frac{1}{2}\,[V(-i) + V^*(-i)] = \gamma + \int_0^{+\infty} \frac{t}{1+t^2}\, dG(t). \tag{6.12}$$

Combining (6.10) and (6.12) we conclude that $\gamma h = 0$ for all $h \in E$ such that (6.11) holds.

Definition 6.3. An operator-valued Stieltjes function $V(z)$ in a finite-dimensional Hilbert space E belongs to the **class** $S(R)$ if in the representation (6.3)

$$\gamma h = 0$$

for all $h \in E$ such that

$$\int_0^{\infty} (dG(t)h, h)_E < \infty. \tag{6.13}$$

We are going to focus though on the subclass $S_0(R)$ of $S(R)$ whose definition is the following.

Definition 6.4. An operator-valued Stieltjes function $V(z)$ in a finite-dimensional Hilbert space E belongs to the **class** $S_0(R)$ if in the representation (6.3) we have

$$\int_0^\infty (dG(t)h,h)_E = \infty, \tag{6.14}$$

for all non-zero $h \in E$.

An L-system Θ of the form (2.3) is called an **accretive L-system** if its operator $\mathbb{A}$ is accretive. The following theorem is the direct realization theorem for the functions of the class $S_0(R)$.

Theorem 6.5. *Let Θ be an accretive L-system of the form* (2.3) *with an invertible channel operator K and a densely defined symmetric operator $\dot{A}$. Then its impedance function $V_\Theta(z)$ of the form* (2.7) *belongs to the class $S_0(R)$.*

Proof. Since our L-system Θ is accretive, then by Theorem 6.2, $V_\Theta(z)$ is a Stieltjes function. Now let us show that $V_\Theta(z)$ belongs to $S_0(R)$. It follows from Theorem 7 of [14] that $E_1 = K^{-1}\mathfrak{L}$, where $\mathfrak{L} = \mathcal{H} \ominus \overline{\mathrm{Dom}(\dot{A})}$ and

$$E_1 = \left\{ h \in E : \int_0^{+\infty} (dG(t)h,h)_E < \infty \right\}.$$

But $\overline{\mathrm{Dom}(\dot{A})} = \mathcal{H}$ and consequently $\mathfrak{L} = \{0\}$. Next, $E_1 = \{0\}$,

$$\int_0^\infty (dG(t)h,h)_E = \infty,$$

for all non-zero $h \in E$, and therefore $V_\Theta(z) \in S_0(R)$. □

We can also state and prove the following inverse realization theorem for the classes $S_0(R)$.

Theorem 6.6. *Let an operator-valued function $V(z)$ belong to the class $S_0(R)$. Then $V(z)$ can be realized as an impedance function of a minimal accretive L-system Θ of the form* (2.3) *with an invertible channel operator K, a densely defined non-negative symmetric operator $\dot{A}$,* $\mathrm{Dom}(T) \neq \mathrm{Dom}(T^*)$, *and a preassigned direction operator J for which $I + iV(-i)J$ is invertible.*[3]

Proof. We have already noted that the class of Stieltjes function lies inside the wider class of all Herglotz-Nevanlinna functions. Thus all we actually have to show is that $S_0(R) \subset N_0(R)$, where the subclass $N_0(R)$ was defined in [16], and that the realizing L-system in the proof of Theorem 11 of [16] appears to be an accretive L-system. The former is rather obvious and follows directly from the definition of the class $S_0(R)$. To see that the realizing L-system is accretive we need to recall that the model L-system Θ was constructed in the proof of Theorem 11 of [16] and

[3]It was shown in [14] that if $J = I$ this invertibility condition is satisfied automatically.

then it was shown that $V_\Theta(z) = V(z)$. But $V(z)$ is a Stieltjes function and hence so is $V_\Theta(z)$. Applying Theorem 6.2 yields the desired result. □

Let us define a subclass of the class $S_0(R)$.

Definition 6.7. An operator-valued Stieltjes function $V(z)$ of the class $S_0(R)$ is said to be a member of the **class** $S_0^K(R)$ if

$$\int_0^\infty \frac{(dG(t)h,h)_E}{t} = \infty, \tag{6.15}$$

for all non-zero $h \in E$.

Below we state and prove direct and inverse realization theorem for this subclass.

Theorem 6.8. *Let Θ be an accretive L-system of the form* (2.3) *with an invertible channel operator K and a densely defined symmetric operator $\dot{A}$. If the Kreĭn-von Neumann extension A_K is a quasi-kernel for* $\mathrm{Re}\,\mathbb{A}$, *then the impedance function $V_\Theta(z)$ of the form* (2.7) *belongs to the class $S_0^K(R)$.*

Conversely, if $V(z) \in S_0^K(R)$, then it can be realized as the impedance function of an accretive L-system Θ of the form (2.3) *with* $\mathrm{Re}\,\mathbb{A}$ *containing A_K as a quasi-kernel and a preassigned direction operator J for which $I + iV(-i)J$ is invertible.*

Proof. We begin with the proof of the second part. First we use realization Theorem 2.4 and Theorem 6.6 to construct a minimal model L-system Θ whose impedance function is $V(z)$. Then we will show that (6.15) is equivalent to the fact that self-adjoint operator A introduced in the proof of Theorem 2.4 (see [14]) and constructed to be a quasi-kernel for $\mathrm{Re}\,\mathbb{A}$, coincides with A_K, that is the Kreĭn-von Neumann extension of the model symmetric operator $\dot{A}$ of multiplication by an independent variable (see [14]). Let $L^2_G(E)$ be a model space constructed in the proof of Theorem (2.4) (see [14]). Let also $E(s)$ be the orthoprojection operator in $L^2_G(E)$ defined by

$$E(s)f(t) = \begin{cases} f(t), & 0 \le t \le s \\ 0, & t > s \end{cases} \tag{6.16}$$

where $f(t) \in C_{00}(E, [0, +\infty))$. Here be $C_{00}(E, [0, +\infty))$ is the set of continuous compactly supported functions $f(t)$, $([0 < t < +\infty))$ with values in E. Then for the operator A, that is the operator of multiplication by independent variable defined in the proof of Theorem 2.4 (see [14]), we have

$$A = \int_0^\infty s\, dE(s), \tag{6.17}$$

and $E(s)$ is the resolution of identity of the operator A. By construction provided in the proof of Theorem 2.4, the operator A is the quasi-kernel of $\mathrm{Re}\,\mathbb{A}$, where $\mathbb{A}$

is an accretive (*)-extension of the model system. Let us calculate $(E(s)f(t),f(t))$ and $(Af(t),f(t))$ (here we use $L^2_G(E)$ scalar product).

$$(E(s)f(t),f(t)) = \int\limits_0^\infty (dG(t)E(s)f(t),f(t))_E = \int\limits_0^s (dG(t)f(t),f(t))_E, \qquad (6.18)$$

$$(Af(t),f(t)) = \int\limits_0^\infty s\,d\left\{\int\limits_0^s (dG(t)f(t),f(t))_E\right\} = \int\limits_0^\infty s\,d(G(s)x(s),x(s))_E. \quad (6.19)$$

The equality $A = A_K$ holds (see Proposition 3.2) if for all $\varphi \in \mathfrak{N}_{-a}$, $\varphi \neq 0$

$$\int_0^\infty \frac{(dE(t)\varphi,\varphi)}{t} = \infty, \qquad (6.20)$$

where $\mathfrak{N}_{-a}$ is the deficiency subspace of the operator $\dot{A}$ corresponding to the point $(-a)$, $(a>0)$. But according to Theorem 2.4 we have

$$\mathfrak{N}_z = \left\{\frac{h}{t-z} \in L^2_G(E) \mid h \in E\right\},$$

and hence

$$\mathfrak{N}_{-a} = \left\{\frac{h}{t+a} \in L^2_G(E) \mid h \in E\right\}. \qquad (6.21)$$

Taking into account (6.15) we have for all $h \in E$

$$\int\limits_0^\infty \frac{(dE(s)\varphi,\varphi)_{L^2_G(E)}}{s} = \int\limits_0^\infty \frac{(dE(s)\frac{h}{t+a},\frac{h}{t+a})_{L^2_G(E)}}{s} = \int\limits_0^\infty \frac{(dG(s)h,h)_E}{s(s+a)^2}.$$

Hence the operator $A = A_K$ iff

$$\int\limits_0^\infty \frac{(dG(t)h,h)_E}{t(t+a)^2} = \infty, \quad \forall h \in E,\ h \neq 0. \qquad (6.22)$$

Let us transform (6.15)

$$\begin{aligned}\int_0^\infty \frac{(dG(t)h,h)_E}{t} &= \int_0^\infty \frac{(t+a)^2}{t}\left(dG(t)\frac{h}{t+a},\frac{h}{t+a}\right)_E \\ &= \int_0^\infty t\left(dG(t)\frac{h}{t+a},\frac{h}{t+a}\right)_E + 2a\int_0^\infty \left(dG(t)\frac{h}{t+a},\frac{h}{t+a}\right)_E \qquad (6.23)\\ &\quad + a^2\int_0^\infty \frac{(dG(t)h,h)_E}{t(t+a)^2}.\end{aligned}$$

Since $\mathrm{Re}\,\mathbb{A}$ is a non-negative self-adjoint bi-extension of $\dot{A}$ in the model system, then we can apply Theorem 4.8 to get (4.9). Then first two integrals in (6.23) converge for a fixed a because of (4.9) and equality

$$\int_0^\infty \left(dG(t)\frac{h}{t+a},\frac{h}{t+a}\right)_E = \int_0^\infty d\,(E(t)\varphi,\varphi)\,, \quad \varphi \in \mathfrak{N}_{-a}.$$

Therefore the divergence of integral in (6.15) completely depends on divergence of the last integral in (6.23).

Now we can prove the first part of the theorem. Let Θ be our L-system with A_K that is a quasi-kernel for $\mathrm{Re}\,\mathbb{A}$, and the impedance function $V_\Theta(z)$. Without loss of generality we can consider Θ as a minimal system, otherwise we would take the principal part of Θ that is minimal and has the same impedance function (see [14]). Furthermore, $V_\Theta(z)$ can be realized as an impedance function of the model L-system Θ_1 constructed in the proof of Theorem 2.4. Some of the elements of Θ_1 were already described above during the proof of the second part of the theorem. If the L-system Θ_1 is not minimal, we consider its principal part $\Theta_{1,0}$ that is described in Theorem 12 of [14] and has the same impedance function as Θ_1. Since both Θ and $\Theta_{1,0}$ share the same impedance function $V_\Theta(z)$ they also have the same transfer function $W_\Theta(z)$ and thus we can apply the theorem on bi-unitary equivalence of [11]. According to this theorem the quasi-kernel operator A_0 of $\Theta_{1,0}$ is unitary equivalent to the quasi-kernel A_K in Θ. Consequently, property (6.20) of A_K gets transferred by the unitary equivalence mapping to the corresponding property of A_0 making it, by Proposition 3.2, the Kreĭn-von Neumann self-adjoint extension of the corresponding symmetric operator $\dot{A}_0$ of $\Theta_{1,0}$. But this implies that the quasi-kernel operator A of Θ_1 (defined by (6.17)) is also the Kreĭn-von Neumann self-adjoint extension and hence has property (6.20) that causes (6.22). Using (6.22) in conjunction with (6.23) we obtain (6.15). That proves the theorem. $\square$

7. Realization of inverse Stieltjes functions

Definition 7.1. We will call an operator-valued Herglotz-Nevanlinna function $V(z)$ in a finite-dimensional Hilbert space E by an **inverse Stieltjes** if $V(z)$ it is holomorphic in $\mathrm{Ext}[0,+\infty)$ and

$$\frac{\mathrm{Im}[V(z)/z]}{\mathrm{Im}\ z} \geq 0. \tag{7.1}$$

Combining (7.1) with (1.1) we obtain (see [18])

$$\sum_{k,l=1}^{n}\left(\frac{V(z_k)/z_k - V(\bar{z}_l)/\bar{z}_l}{z_k-\bar{z}_l}h_k, h_l\right)_E \geq 0,$$

for an arbitrary sequence $\{z_k\}$ $(k=1,\dots,n)$ of $(\mathrm{Im}\, z_k>0)$ complex numbers and a sequence of vectors $\{h_k\}$ in E. It can be shown (see [23]) that every inverse Stieltjes function $V(z)$ in a finite-dimensional Hilbert space E admits the following integral representation

$$V(z) = \alpha + z\beta + \int_0^\infty \left(\frac{1}{t-z} - \frac{1}{t}\right) dG(t), \tag{7.2}$$

where $\alpha \leq 0$, $\beta \geq 0$, and $G(t)$ is a non-decreasing on $[0,+\infty)$ operator-valued function such that

$$\int_0^\infty \frac{(dG(t)h,h)}{t+t^2} < \infty, \quad \forall h \in E.$$

The following definition provides the description of all realizable inverse Stieltjes operator-valued functions.

Definition 7.2. An operator-valued inverse Stieltjes function $V(z)$ in a finite-dimensional Hilbert space E is a member of the **class** $S^{-1}(R)$ if in the representation (7.2) we have

$$\text{i)} \qquad \beta = 0,$$

$$\text{ii)} \qquad \alpha h = 0,$$

for all $h \in E$ with

$$\int_0^\infty (dG(t)h, h)_E < \infty.$$

In what follows we will, however, be mostly interested in the following subclass of $S^{-1}(R)$.

Definition 7.3. An inverse Stieltjes function $V(z) \in S^{-1}(R)$ is a member of the **class** $S_0^{-1}(R)$ if

$$\int_0^\infty (dG(t)h, h)_E = \infty,$$

for all $h \in E$, $h \neq 0$.

We recall that an L-system Θ of the form (2.3) is called **accumulative** if its state-space operator $\mathbb{A}$ is accumulative, i.e., satisfies (2.5). It is easy to see that if an L-system is accumulative, then (2.5) implies that the operator $\dot{A}$ of the system is non-negative and both operators T and T^* are accretive.

The following statement is the direct realization theorem for the functions of the class $S_0^{-1}(R)$.

Theorem 7.4. *Let Θ be an accumulative L-system of the form* (2.3) *with an invertible channel operator K and $\overline{\mathrm{Dom}(\dot{A})} = \mathcal{H}$. Then its impedance function $V_\Theta(z)$ of the form* (2.7) *belongs to the class $S_0^{-1}(R)$.*

Proof. First we will show that $V_\Theta(z)$ is an inverse Stieltjes function. Let $\{z_k\}$ $(k = 1, \dots, n)$ is a sequence of non-real $(z_k \neq \bar{z}_k)$ complex numbers and φ_k $(z_k \neq \bar{z}_k)$ is a sequence of elements of $\mathfrak{N}_{z_k}$, the defect subspace of the operator $\dot{A}$. Then for every k there exists $h_k \in E$ such that

$$\varphi_k = z_k(\mathrm{Re}\,\mathbb{A} - z_k I)^{-1} K h_k, \qquad (k = 1, \dots, n). \tag{7.3}$$

Taking into account that $\dot{A}^* \varphi_k = z_k \varphi_k$, formula (7.3), and letting $\varphi = \sum_{k=1}^n \varphi_k$ we get

$$\begin{aligned}
&(\dot{A}^*\varphi, \varphi) + (\varphi, \dot{A}^*\varphi) - (\mathrm{Re}\,\mathbb{A}\varphi, \varphi) \\
&= \sum_{k,l=1}^n \Big[(\dot{A}^*\varphi_k, \varphi_l) + (\varphi_k, \dot{A}^*\varphi_l) - (\mathrm{Re}\,\mathbb{A}\varphi_k, \varphi_l)\Big] \\
&= \sum_{k,l=1}^n ([-\mathrm{Re}\,\mathbb{A} + z_k + \bar{z}_l]\varphi_k, \varphi_l)
\end{aligned}$$

$$= \sum_{k,l=1}^{n} \Bigg(\frac{(\mathrm{Re}\,\mathbb{A} - \bar{z}_l I)^{-1}(\bar{z}_l(\mathrm{Re}\,\mathbb{A} - \bar{z}_l I) - z_k(\mathrm{Re}\,\mathbb{A} - z_k I))(\mathrm{Re}\,\mathbb{A} - z_k I)^{-1}}{z_k \bar{z}_l (z_k - \bar{z}_l)}$$

$$\times K h_k, K h_l \Bigg)$$

$$= \sum_{k,l=1}^{n} \left(\frac{\bar{z}_l K^*(\mathrm{Re}\,\mathbb{A} - z_k I)^{-1} K - z_k K^*(\mathrm{Re}\,\mathbb{A} - z_l I)^{-1} K}{z_k \bar{z}_l (z_k - \bar{z}_l)} h_k, h_l \right)$$

$$= \sum_{k,l=1}^{n} \left(\frac{\bar{z}_l V_\Theta(z_k) - z_k V_\Theta(\bar{z}_l)}{z_k z_l (z_k - \bar{z}_l)} h_k, h_l \right) \geq 0.$$

The last line can be re-written as follows

$$\sum_{k,l=1}^{n} \left(\frac{V_\Theta(z_k)/z_k - V_\Theta(\bar{z}_l)/\bar{z}_l}{z_k - \bar{z}_l} h_k, h_l \right) \geq 0. \tag{7.4}$$

Letting in (7.4) $n = 1$, $z_1 = z$, and $h_1 = h$ we get

$$\left(\frac{V_\Theta(z)/z - V_\Theta(\bar{z})/\bar{z}}{z - \bar{z}} h, h \right) \geq 0, \tag{7.5}$$

which means

$$\frac{\mathrm{Im}\,(V_\Theta(z)/z)}{\mathrm{Im}\, z} \geq 0,$$

and therefore $V_\Theta(z)/z$ is a Herglotz-Nevanlinna function. In Theorem 8 of [14] we have shown that $V_\Theta(z) \in N(R)$. Applying (7.1) we conclude that $V_\Theta(z)$ is an inverse Stieltjes function.

Now we will show that $V_\Theta(z)$ belongs to $S^{-1}(R)$. As any inverse Stieltjes function $V_\Theta(z)$ has its integral representation (7.2) where $\alpha \leq 0$, $\beta \geq 0$, and

$$\int_0^\infty \frac{(dG(t)h, h)}{t + t^2} < \infty, \quad \forall h \in E.$$

In a neighborhood of zero the expression $(t + t^2)$ is equivalent to the $(t + t^3)$ and in a neighborhood of the point at infinity

$$\frac{1}{t + t^3} < \frac{1}{t + t^2}.$$

Hence,

$$\int_0^\infty \frac{(dG(t)h, h)}{t + t^3} < \infty, \quad \forall h \in E.$$

Furthermore,

$$V_\Theta(z) = \alpha + z\beta + \int_0^\infty \left(\frac{1}{t - z} - \frac{t}{1 + t^2} + \frac{t}{1 + t^2} - \frac{1}{t} \right) dG(t)$$

$$= \left(\alpha - \int_0^\infty \frac{dG(t)}{t + t^3} \right) + z\beta + \int_0^\infty \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) dG(t).$$

On the other hand, as it was shown in [14], a Herglotz-Nevanlinna function can be realized if and only if it belongs to the class $N(R)$ and hence in representation (1.1) condition (1.7) holds. Considering this and the uniqueness of the function $G(t)$ we obtain

$$\left(\alpha - \int_0^\infty \frac{dG(t)}{t+t^3}\right) f = \int_0^{+\infty} \frac{t}{1+t^2} dG(t)f, \tag{7.6}$$

for all $f \in E$ such that $\int_{-\infty}^{+\infty} (dG(t)f, f)_E < \infty$. Solving (7.6) for α we get

$$\alpha f = \int_0^\infty \frac{1}{t}\, dG(t)f, \tag{7.7}$$

for the same selection of f. The left-hand side of (7.7) is non-positive but the right-hand side is non-negative. This means that $\alpha = 0$ and $V_\Theta(z) \in S^{-1}(R)$. The proof of the fact that $V_\Theta(z) \in S_0^{-1}(R)$ is similar to the proof of Theorem 6.5. □

The inverse realization theorem can be stated and proved for the class $S_0^{-1}(R)$ as follows.

Theorem 7.5. *Let an operator-valued function $V(z)$ belong to the class $S_0^{-1}(R)$. Then $V(z)$ can be realized as an impedance function of an accumulative minimal L-system Θ of the form* (2.3) *with an invertible channel operator K, a non-negative densely defined symmetric operator $\dot{A}$ and $J = I$.*

Proof. The class $S_0^{-1}(R)$ is a subclass of $N_0(R)$ and hence it is realizable by a minimal L-system Θ with a densely defined symmetric operator $\dot{A}$ and $J = I$. Thus all we have to show is that the L-system Θ we have constructed in the proof of Theorem 11 of [16] is an accumulative L-system, i.e., satisfying the condition (2.5).

Since the L-system Θ is minimal then the operator $\dot{A}$ is prime. Applying (6.8) yields

$$\underset{z\neq\bar{z}}{c.l.s.}\, \mathfrak{N}_z = \mathcal{H}, \quad z \neq \bar{z}. \tag{7.8}$$

In the proof of Theorem 7.4 we have shown that

$$(\mathrm{Re}\, \mathbb{A}\varphi, \varphi) \leq (\dot{A}^*\varphi, \varphi) + (\varphi, \dot{A}^*\varphi), \quad \varphi = \sum_{k=1}^n \varphi_k, \quad \varphi_k \in \mathfrak{N}_{z_k}, \tag{7.9}$$

is equivalent to (7.4), where z_k are defined by (7.3). Combining (7.8) and (7.9) we get property (2.5) and conclude that Θ is an accumulative L-system. □

It is not hard to see that members of the classes $S_0(R)$ and $S_0^{-1}(R)$ are the Kreĭn-Langer Q-functions [27] corresponding to a self-adjoint extensions of a densely defined symmetric operator.

Now we define a subclass of the class $S_0^{-1}(R)$.

Definition 7.6. An operator-valued Stieltjes function $V(z)$ of the class $S_0^{-1}(R)$ is said to be a member of the **class** $S_{0,F}^{-1}(R)$ if

$$\int_0^\infty \frac{t}{t^2+1}\,(dG(t)h,h)_E = \infty, \tag{7.10}$$

for all non-zero $h \in E$.

Theorem 7.7. *Let Θ be an accumulative L-system of the form* (2.3) *with an invertible channel operator K and a symmetric densely defined operator $\dot{A}$. If Friedrichs extension A_F is a quasi-kernel for* $\mathrm{Re}\,\mathbb{A}$*, then the impedance $V_\Theta(z)$ of the form* (2.7) *belongs to the class $S_{0,F}^{-1}(R)$.*

Conversely, if $V(z) \in S_{0,F}^{-1}(R)$, then it can be realized as an impedance of an accumulative L-system Θ of the form (2.3) *with* $\mathrm{Re}\,\mathbb{A}$ *containing A_F as a quasi-kernel and a preassigned direction operator J for which $I + iV(-i)J$ is invertible.*

Proof. Following the framework of the proof of Theorem 6.8, we begin with the proof of the second part. First we use the realization Theorem 2.4 and Theorem 7.5 to construct a minimal model L-system Θ whose impedance function is $V(z)$. Then we will show that (6.15) is equivalent to the fact that self-adjoint operator A introduced in the proof of Theorem (2.4) (see [14]) and constructed to be a quasi-kernel for $\mathrm{Re}\,\mathbb{A}$, coincides with A_F, that is the Friedrichs extension of the symmetric operator $\dot{A}$ of multiplication by an independent variable (see [14]). Let $L^2_G(E)$ be a model space constructed in the proof or of Theorem (2.4). Let also $E(s)$ be the orthoprojection operator in $L^2_G(E)$ defined by (6.16). Then for the operator A defined in the proof of Theorem 2.4 (see [14]) we have

$$A = \int_0^\infty t\,dE(t),$$

and $E(t)$ is the spectral function of operator A. As we have shown in the proof of Theorem 6.8 the relations (6.18) and (6.19) take place. The equality $A = A_F$ holds (see Proposition 3.2) if for all $\varphi \in \mathfrak{N}_{-a}$

$$\int_0^\infty t\,(dE(t)\varphi,\varphi)_E = \infty, \tag{7.11}$$

where $\mathfrak{N}_{-a}$ is the deficiency subspace of the operator $\dot{A}$ corresponding to the point $(-a)$, $(a > 0)$. But according to Theorem 2.4 we have $\mathfrak{N}_{-a}$ described by (6.21). Taking into account (7.10) we have for all $h \in E$

$$\int\limits_0^\infty s(dE(s)\varphi,\varphi)_{L^2_G(E)} = \int\limits_0^\infty sd\left(E(s)\frac{h}{t+a},\frac{h}{t+a}\right)_{L^2_G(E)} = \int\limits_0^\infty \frac{s\,(dG(s)h,h)_E}{(s+a)^2}.$$

Hence the operator $A = A_F$ iff

$$\int\limits_0^\infty \frac{t\,(dG(t)h,h)_E}{(t+a)^2} = \infty, \quad \forall h \in E,\ h \neq 0. \tag{7.12}$$

Let us transform (7.10)

$$
\begin{aligned}
\int_0^\infty \frac{t}{t^2+1}\,(dG(t)h,h)_E &= \int_0^\infty \frac{t(t+a)^2}{t^2+1}\left(dG(t)\frac{h}{t+a},\frac{h}{t+a}\right)_E\\
&= \int_0^\infty \frac{t}{(t+a)^2}\cdot\frac{t^2}{t^2+1}\left(dG(t)\frac{h}{t+a},\frac{h}{t+a}\right)_E\\
&\quad + 2a\int_0^\infty \frac{t^2}{(t+a)^2(t^2+1)}\left(dG(t)\frac{h}{t+a},\frac{h}{t+a}\right)_E\\
&\quad + a^2\int_0^\infty \frac{1}{t^2+1}\cdot\frac{t\,(dG(t)h,h)_E}{(t+a)^2}.
\end{aligned}
\tag{7.13}
$$

Consider the following obvious inequality

$$
\frac{t^2}{(t+a)^2(t^2+1)}-\frac{1}{t^2+1}=\frac{t^2-(t+a)^2}{(t+a)^2(t^2+1)}=\frac{(2t+a)(-a)}{(t+a)^2(t^2+1)}<0.
$$

Taking into account this inequality and the fact that the integral

$$
\int_0^\infty \frac{(dG(t)h,h)_E}{t^2+1},
$$

converges for all $h\in E$, we conclude that the second integral in (7.13) is convergent. Let us denote this integral as Q. Then using (7.13) and obvious estimates we obtain

$$
\begin{aligned}
\int_0^\infty \frac{t}{t^2+1}\,(dG(t)h,h)_E \le \int_0^\infty \frac{t}{(t+a)^2}&\left(dG(t)\frac{h}{t+a},\frac{h}{t+a}\right)_E\\
&+2aQ+a^2\int_0^\infty \frac{t\,(dG(t)h,h)_E}{(t+a)^2},
\end{aligned}
$$

or

$$
\int_0^\infty \frac{t}{t^2+1}\,(dG(t)h,h)_E \le (a^2+1)\int_0^\infty \frac{t}{(t+a)^2}\left(dG(t)\frac{h}{t+a},\frac{h}{t+a}\right)_E+2aQ.
$$

Since $V(z)\in S_0^{-1}(R)$, then (7.7) holds and the integral on the left diverges causing the integral on the right side diverge as well. Thus $A=A_F$.

Now we can prove the first part of the theorem. Let Θ be our L-system with A_F that is a quasi-kernel for $\mathrm{Re}\,\mathbb{A}$, and the impedance function $V_\Theta(z)$. Then $V_\Theta(z)$ can be realized as an impedance function of the model L-system Θ_1 constructed in the proof of Theorem 2.4. Repeating the argument of the second part of the proof of Theorem 6.8 with A_K replaced by A_F we conclude that the quasi-kernel operator A of Θ_1 is the Friedrichs self-adjoint extension and hence has property (7.11) that in turn causes (7.12) for any $a>0$. Let $a=1$, then by (7.12)

$$
\infty=\int_0^\infty \frac{t\,(dG(t)h,h)_E}{(t+1)^2}\le\int_0^\infty \frac{t\,(dG(t)h,h)_E}{t^2+1},\quad \forall h\in E,\ h\neq 0,
$$

and hence the integral on the right diverges and (7.10) holds. This completes the proof. □

8. Examples

Let $\mathcal{H} = L_2[a, +\infty)$ and $l(y) = -y'' + q(x)y$ where q is a real locally summable function. Suppose that the symmetric operator

$$\left\{ \begin{array}{l} Ay = -y'' + q(x)y \\ y(a) = y'(a) = 0 \end{array} \right. \tag{8.1}$$

has deficiency indices (1,1). Let D^* be the set of functions locally absolutely continuous together with their first derivatives such that $l(y) \in L_2[a, +\infty)$. Consider $\mathcal{H}_+ = D(A^*) = D^*$ with the scalar product

$$(y, z)_+ = \int_a^\infty \left(y(x)\overline{z(x)} + l(y)\overline{l(z)} \right) dx, \quad y,\ z \in D^*.$$

Let $\mathcal{H}_+ \subset L_2[a, +\infty) \subset \mathcal{H}_-$ be the corresponding triplet of Hilbert spaces. Consider operators

$$\left\{ \begin{array}{l} T_h y = l(y) = -y'' + q(x)y \\ hy(a) = y'(a) \end{array} \right. , \quad \left\{ \begin{array}{l} T_h^* y = l(y) = -y'' + q(x)y \\ \overline{h}y(a) = y'(a) \end{array} \right. , \tag{8.2}$$

$$\left\{ \begin{array}{l} \widehat{A}y = l(y) = -y'' + q(x)y \\ \mu y(a) = y'(a) \end{array} \right. , \quad \operatorname{Im}\mu = 0.$$

It is well known [1] that $\widehat{A} = \widehat{A^*}$. The following theorem was proved in [11].

Theorem 8.1. *The set of all* $(*)$*-extensions of a non-self-adjoint Schrödinger operator* T_h *of the form* (8.2) *in* $L_2[a, +\infty)$ *can be represented in the form*

$$\begin{aligned} \mathbb{A}y &= -y'' + q(x)y - \frac{1}{\mu - h}\,[y'(a) - hy(a)]\,[\mu\delta(x-a) + \delta'(x-a)], \\ \mathbb{A}^*y &= -y'' + q(x)y - \frac{1}{\mu - \overline{h}}\,[y'(a) - \overline{h}y(a)]\,[\mu\delta(x-a) + \delta'(x-a)]. \end{aligned} \tag{8.3}$$

In addition, the formulas (8.3) *establish a one-to-one correspondence between the set of all* $(*)$*-extensions of a Schrödinger operator* T_h *of the form* (8.2) *and all real numbers* $\mu \in [-\infty, +\infty]$.

Suppose that the symmetric operator A of the form (8.1) with deficiency indices (1,1) is nonnegative, i.e., $(Af, f) \geq 0$ for all $f \in D(A)$). It was shown in [34] that the Schrödinger operator T_h of the form (8.2) is accretive if and only if

$$\operatorname{Re} h \geq -m_\infty(-0), \tag{8.4}$$

where $m_\infty(\lambda)$ is the Weyl-Titchmarsh function [1]. For real h such that $h \geq -m_\infty(-0)$ we get a description of all nonnegative self-adjoint extensions of an operator A. For $h = -m_\infty(-0)$ the corresponding operator

$$\left\{ \begin{array}{l} A_K\, y = -y'' + q(x)y \\ y'(a) + m_\infty(-0)y(a) = 0 \end{array} \right. \tag{8.5}$$

is the Kreĭn-von Neumann extension of A and for $h = +\infty$ the corresponding operator

$$\begin{cases} A_F\, y = -y'' + q(x)y \\ y(a) = 0 \end{cases} \tag{8.6}$$

is the Friedrichs extension of A (see [34], [11]).

We conclude this paper with two simple illustrations for Theorems 6.8 and 7.7.

Example. Consider a function

$$V(z) = \frac{i}{\sqrt{z}}. \tag{8.7}$$

A direct check confirms that $V(z)$ in (8.7) is a Stieltjes function. It was shown in [29] that the inversion formula

$$G(t) = C + \lim_{y\to 0} \frac{1}{\pi} \int_0^t \operatorname{Im}\left(\frac{i}{\sqrt{x+iy}}\right) dx \tag{8.8}$$

describes the measure $G(t)$ in the representation (6.3). By direct calculations one can confirm that

$$V(z) = \int_0^\infty \frac{dG(t)}{t-z} = \frac{i}{\sqrt{z}}, \quad \text{and that} \quad \int_0^\infty \frac{dG(t)}{t} = \int_0^\infty \frac{dt}{\pi t^{3/2}} = \infty.$$

Thus we can conclude that $V(z) \in S_0^K(R)$. It was shown in [17] that $V(z)$ can be realized as the impedance function of the L-system

$$\Theta = \begin{pmatrix} \mathbb{A} & K & 1 \\ \mathcal{H}_+ \subset L_2[a,+\infty) \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix},$$

where

$$\mathbb{A}\, y = -y'' + [iy(0) - y'(0)]\delta(x).$$

The operator T_h in this case is

$$\begin{cases} T_h y = -y'' \\ y'(0) = iy(0), \end{cases} \tag{8.9}$$

and channel operator $Kc = cg$, $g = \delta(x)$, $(c \in \mathbb{C})$ with

$$K^* y = (y, g) = y(0).$$

The real part of $\mathbb{A}$

$$\operatorname{Re} \mathbb{A}\, y = -y'' - y'(0)\delta(x)$$

contains the self-adjoint quasi-kernel

$$\begin{cases} \widehat{A} y = -y'' \\ y'(0) = 0. \end{cases}$$

Clearly, $\widehat{A} = A_K$, where A_K is given by (8.5).

Example. Consider a function

$$V(z) = i\sqrt{z}. \tag{8.10}$$

A direct check confirms that $V(z)$ in (8.10) is an inverse Stieltjes function. Applying the inversion formula similar to (8.8) we obtain

$$G(t) = C + \lim_{y\to 0} \frac{1}{\pi} \int_0^t \operatorname{Im}\left(i\sqrt{x+iy}\right) dx,$$

where $G(t)$ is the function in the representation (7.2). By direct calculations one can confirm that

$$V(z) = \int_0^\infty \left(\frac{1}{t-z} - \frac{1}{t}\right) dG(t) = i\sqrt{z},$$

and that

$$\int_0^\infty \frac{t}{t^2+1}\, dG(t) = \int_0^\infty \frac{dG(t)}{t} = \int_0^\infty \frac{dt}{\pi\sqrt{t}} = \infty.$$

Thus we can conclude that $V(z) \in S_{0,F}^{-1}(R)$. It was shown in [18] that $V(z)$ can be realized as the impedance function of the L-system

$$\Theta = \begin{pmatrix} \mathbb{A} & K & 1 \\ \mathcal{H}_+ \subset L_2[a,+\infty) \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix},$$

where

$$\mathbb{A}\, y = -y'' - [iy'(0) + y'(0)]\delta'(x).$$

The operator T_h in this case is again given by (8.9) and channel operator $Kc = cg$, $g = \delta'(x)$, $(c \in \mathbb{C})$ with

$$K^* y = (y, g) = -y'(0).$$

The real part of $\mathbb{A}$

$$\operatorname{Re} \mathbb{A}\, y = -y'' - y(0)\delta'(x)$$

contains the self-adjoint quasi-kernel

$$\begin{cases} \widehat{A} y = -y'' \\ y(0) = 0. \end{cases}$$

Clearly, $\widehat{A} = A_F$, where A_F is given by (8.6).

Acknowledgment. We would like to thank the referee for valuable remarks.

References

[1] Akhiezer, N.I., Glazman, I.M.: Theory of Linear Operators in Hilbert Space. Dover, New York, (1993)

[2] Ando, T., Nishio, K.: Positive Selfadjoint Extensions of Positive Symmetric Operators. Tohóku Math. J., **22**, 65–75 (1970)

[3] Arlinskiĭ, Yu.M.: On inverse problem of the theory of characteristic functions of unbounded operator colligations. Dopovidi Akad. Nauk Ukrain. RSR, Ser. A, No. 2, 105–109 (1976)

[4] Arlinskiĭ, Yu.M.: Regular (∗)-extensions of quasi-Hermitian operators in rigged Hilbert spaces. (Russian), Izv. Akad. Nauk Armyan. SSR, Ser.Mat., **14**, No. 4, 297–312 (1979)

[5] Arlinskiĭ, Yu.M.: On regular (∗)-extensions and characteristic matrix-valued functions of ordinary differential operators. Boundary value problems for differential operators, Kiev, 3–13, (1980)

[6] Arlinskiĭ, Yu.M.: On accretive (*)-extensions of a positive symmetric operator. (Russian), Dokl. Akad. Nauk. Ukraine, Ser. A, No. 11, 3–5 (1980)

[7] Arlinskiĭ, Yu.M., Tsekanovskiĭ, E.R.: Nonselfadjoint contractive extensions of a Hermitian contraction and theorems of Kreĭn. Russ. Math. Surv., **37:1**, 151–152 (1982)

[8] Arlinskiĭ, Yu.M., Tsekanovskiĭ, E.R.: Sectorial extensions of positive Hermitian operators and their resolvents. (Russian), Akad. Nauk. Armyan. SSR, Dokl., **79**, No.5, 199–203 (1984)

[9] Arlinskiĭ, Yu.M., Tsekanovskiĭ, E.R.: Regular (∗)-extension of unbounded operators, characteristic operator-functions and realization problems of transfer mappings of linear systems. Preprint, VINITI, -2867. -79, Dep.-72 (1979)

[10] Yu.M. Arlinskiĭ and E.R. Tsekanovskiĭ: Quasi-self-adjoint contractive extensions of Hermitian contractions, Teor. Funkts., Funkts. Anal. Prilozhen, **50**, 9–16 (1988) (Russian). English translation in J. Math. Sci. **49**, No. 6, 1241–1247 (1990).

[11] Arlinskiĭ, Yu.M., Tsekanovskiĭ, E.R.: Linear systems with Schrödinger operators and their transfer functions. Oper. Theory Adv. Appl., **149**, 47–77 (2004)

[12] Arlinskiĭ, Yu.M., Tsekanovskiĭ, E.R.: The von Neumann problem for nonnegative symmetric operators. Integral Equations Operator Theory, **51**, No. 3, 319-356 (2005)

[13] Belyi, S.V., Hassi, S., de Snoo, H.S.V., Tsekanovskiĭ, E.R.: On the realization of inverse of Stieltjes functions. Proceedings of MTNS-2002, University of Notre Dame, CD-ROM, 11p., (2002)

[14] Belyi, S.V., Tsekanovskiĭ, E.R.: Realization theorems for operator-valued R-functions. Operator theory: Advances and Applications, **98**, Birkhäuser Verlag Basel, 55–91 (1997)

[15] Belyi, S.V., Tsekanovskiĭ, E.R.: Multiplication Theorems for J-contractive operator-valued functions. Fields Institute Communications, **25**, 187–210 (2000)

[16] Belyi, S.V., Tsekanovskiĭ, E.R.: On classes of realizable operator-valued R-functions. Operator theory: Advances and Applications, **115**, Birkhäuser Verlag Basel, (2000), 85–112.

[17] Belyi, S.V., Tsekanovskiĭ, E.R.: Stieltjes like functions and inverse problems for systems with Schrödinger operator. Operators and Matrices, vol. 2, No. 2, 265–296 (2008)

[18] Belyi, S.V., Tsekanovskiĭ, E.R.: Inverse Stieltjes like functions and inverse problems for systems with Schrödinger operator. Operator Theory: Advances and Applications, vol. 197, 21–49 (2009)

[19] Douglas, R.G.: On majorization, factorization and range inclusion of operators in Hilbert space. Proc. Amer. Math. Soc. **17**, 413–416 (1966)

[20] Dovzhenko, I., Tsekanovskiĭ, E.R.: Classes of Stieltjes operator-valued functions and their conservative realizations. Soviet Math. Dokl., **41**, no. 2, 201–204 (1990)

[21] Fillmore, P.A., Williams, J.P.: On operator ranges. Advances in Math. **7**, 254–281 (1971)

[22] Gesztesy, F., Tsekanovskiĭ, E.R.: On Matrix-Valued Herglotz Functions. Math. Nachr. **218**, 61–138 (2000)

[23] Kac, I.S., Kreĭn, M.G.: R-functions – analytic functions mapping the upper half-plane into itself. Amer. Math. Soc. Transl., (2), **103**, 1–18 (1974)

[24] Kato, T.: Perturbation Theory for Linear Operators. Springer-Verlag, (1966)

[25] Kreĭn, M.G.: The Theory of Selfadjoint Extensions of Semibounded Hermitian Transformations and its Applications. I, (Russian) Mat. Sbornik **20**, No. 3, 431–495 (1947)

[26] Kreĭn, M.G.: The Theory of Selfadjoint Extensions of Semibounded Hermitian Transformations and its Applications, II, (Russian) Mat. Sbornik **21**, No. 3, 365–404 (1947)

[27] Kreĭn, M.G., Langer, H.: Über die Q-Function eines Π-Hermiteschen Operators im Raum Π_κ. Acta Sci. Math. Szeged, **34**, 191–230 (1973)

[28] Malamud, M.: On some classes of Hermitian operators with gaps. (Russian) Ukrainian Mat.J., **44**, No. 2, 215–234 (1992)

[29] Naimark, M.A.: Linear Differential Operators II. F. Ungar Publ., New York, (1968)

[30] Okunskiĭ, M.D., Tsekanovskiĭ, E.R.: On the theory of generalized selfadjoint extensions of semibounded operators. (Russian) Funkcional. Anal. i Prilozen., **7**, No. 3, 92–93 (1973)

[31] Phillips, R.: On dissipative operators, in “Lectures in Differential Equations”, vol. II, Van Nostrand-Reinhold, New York, 65–113 (1965).

[32] Tsekanovskiĭ, E.R.: Non-self-adjoint accretive extensions of positive operators and theorems of Friedrichs-Kreĭn-Phillips. Funct. Anal. Appl. **14**, 156–157 (1980)

[33] Tsekanovskiĭ, E.R.: Friedrichs and Kreĭn extensions of positive operators and holomorphic contraction semigroups. Funct. Anal. Appl. **15**, 308–309 (1981)

[34] Tsekanovskiĭ, E.R.: Accretive Extensions and Problems on Stieltjes Operator-Valued Functions Relations. Operator Theory: Advan. and Appl., **59**, 328–347 (1992)

[35] Tsekanovskiĭ, E.R., Šmuljan, Yu.L.: The theory of bi-extensions of operators on rigged Hilbert spaces. Unbounded operator colligations and characteristic functions. Russ. Math. Surv., **32**, 73–131 (1977)

Yury Arlinskiĭ
Department of Mathematics
East Ukrainian National University
Kvartal Molodyozhny, 20-A
91034 Lugansk, Ukraine
e-mail: yma@snu.edu.ua

Sergey Belyi
Department of Mathematics
Troy University
Troy, AL 36082, USA
e-mail: sbelyi@troy.edu

Eduard Tsekanovskiĭ
Department of Mathematics
Niagara University
New York 14109, USA
e-mail: tsekanov@niagara.edu