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Inverse Stieltjes-like Functions and Inverse Problems for Systems with Schrödinger Operator

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To the memory of Moshe Livšic, a remarkable human being and a great mathematician

Abstract. A class of scalar inverse Stieltjes-like functions is realized as linear-fractional transformations of transfer functions of conservative systems based on a Schrödinger operator T_h in $L_2[a, +\infty)$ with a non-selfadjoint boundary condition. In particular it is shown that any inverse Stieltjes function of this class can be realized in the unique way so that the main operator $\mathbb{A}$ possesses a special semi-boundedness property. We derive formulas that restore the system uniquely and allow to find the exact value of a non-real boundary parameter h of the operator T_h as well as a real parameter μ that appears in the construction of the elements of the realizing system. An elaborate investigation of these formulas shows the dynamics of the restored parameters h and μ in terms of the changing free term α from the integral representation of the realizable function.

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1. Introduction

The role of realizations of different classes of holomorphic operator-valued functions is universally recognized in the spectral analysis of non-self-adjoint operators, interpolation problems, and system theory, with the attention to them growing over the years. The literature on realization theory is too extensive to be discussed thoroughly in this paper. We refer a reader, however, to [2], [3], [7], [8], [9], [10], [11], [12], [20], [27], [26], and the literature therein. This paper is the second in a series where we study realizations of a subclass of Herglotz-Nevanlinna functions with the systems based upon a Schrödinger operator. In [14] we have considered a class of scalar Stieltjes-like functions. Here we focus our attention on another important subclass of Herglotz-Nevanlinna functions, the so-called inverse Stieltjes-like functions.

We recall that an operator-valued function $V(z)$ acting on a finite-dimensional Hilbert space E belongs to the class of operator-valued Herglotz-Nevanlinna functions if it is holomorphic on $\mathbb{C}\setminus\mathbb{R}$, if it is symmetric with respect to the real axis, i.e., $V(z)^*=V(\bar z)$, $z\in\mathbb{C}\setminus\mathbb{R}$, and if it satisfies the positivity condition

$$\operatorname{Im} V(z)\geq 0,\quad z\in\mathbb{C}_+.$$

It is well known (see, e.g., [18], [19]) that operator-valued Herglotz-Nevanlinna functions admit the following integral representation:

$$V(z)=Q+Lz+\int_{\mathbb{R}}\left(\frac{1}{t-z}-\frac{t}{1+t^2}\right)dG(t),\quad z\in\mathbb{C}\setminus\mathbb{R}, \tag{1.1}$$

where $Q=Q^*$, $L\geq 0$, and $G(t)$ is a nondecreasing operator-valued function on $\mathbb{R}$ with values in the class of nonnegative operators in E such that

$$\int_{\mathbb{R}}\frac{(dG(t)x,x)_E}{1+t^2}<\infty,\quad x\in E. \tag{1.2}$$

The realization of a selected class of Herglotz-Nevanlinna functions is provided by a linear conservative system Θ of the form

$$\begin{cases}(\mathbb{A}-zI)x=KJ\varphi_-\\ \varphi_+=\varphi_--2iK^*x\end{cases} \tag{1.3}$$

or

$$\Theta=\begin{pmatrix}\mathbb{A} & K & J\\ \mathcal{H}_+\subset\mathcal{H}\subset\mathcal{H}_- & & E\end{pmatrix}. \tag{1.4}$$

In this system $\mathbb{A}$, the *main operator* of the system, is a so-called $(*)$-extension, which is a bounded linear operator from $\mathcal{H}_+$ into $\mathcal{H}_-$ extending a symmetric operator A in $\mathcal{H}$, where $\mathcal{H}_+\subset\mathcal{H}\subset\mathcal{H}_-$ is a rigged Hilbert space. Moreover, K is a bounded linear operator from the finite-dimensional Hilbert space E into $\mathcal{H}_-$, while $J=J^*=J^{-1}$ is acting on E, are such that $\operatorname{Im}\mathbb{A}=KJK^*$. Also, $\varphi_-\in E$ is an input vector, $\varphi_+\in E$ is an output vector, and $x\in\mathcal{H}_+$ is a vector of the state space of the system Θ. The system described by (1.3)–(1.4) is called a rigged canonical system of the *Livšic type* [24] or (in operator theory) the *Brodskiĭ-Livšic rigged operator colligation*, cf., e.g., [11], [12], [15]. The operator-valued function

$$W_\Theta(z)=I-2iK^*(\mathbb{A}-zI)^{-1}KJ \tag{1.5}$$

is a transfer function (or characteristic function) of the system Θ. It was shown in [11] that an operator-valued function $V(z)$ acting on a Hilbert space E of the form (1.1) can be represented and realized in the form

$$V(z)=i[W_\Theta(z)+I]^{-1}[W_\Theta(z)-I]=K^*(\mathbb{A}_R-zI)^{-1}K, \tag{1.6}$$

where $W_\Theta(z)$ is a transfer function of some canonical scattering ($J = I$) system Θ, and where the "*real part*" $\mathbb{A}_R = \frac{1}{2}(\mathbb{A} + \mathbb{A}^*)$ of $\mathbb{A}$ satisfies $\mathbb{A}_R \supset \hat{A} = \hat{A}^* \supset A$ if and only if the function $V(z)$ in (1.1) satisfies the following two conditions:

$$\begin{cases} L = 0, \\ Qx = \int_{\mathbb{R}} \frac{t}{1+t^2}\, dG(t)x \quad \text{when} \quad \int_{\mathbb{R}} (dG(t)x, x)_E < \infty. \end{cases} \tag{1.7}$$

In the current paper we specialize in an important subclass of Herglotz-Nevanlinna functions, the class of inverse Stieltjes-like functions that also includes inverse Stieltjes functions (see [13]). In Section 4 we specify a subclass of realizable inverse Stieltjes operator-functions and show that any member of this subclass can be realized by a system of the form (1.4) whose main operator $\mathbb{A}$ satisfies inequality

$$(\mathbb{A}_R f, f) \le (A^* f, f) + (f, A^* f), \quad f \in \mathcal{H}_+.$$

In Section 5 we introduce a class of scalar inverse Stieltjes-like functions. Then we rely on the general realization results developed in Section 4 (see also [13] and [14]) to restore a system Θ of the form (1.4) containing the Schrödinger operator in $L_2[a, +\infty)$ with non-self-adjoint boundary conditions

$$\begin{cases} T_h y = -y'' + q(x)y \\ y'(a) = hy(a) \end{cases}, \quad \left(q(x) = \overline{q(x)},\ \operatorname{Im} h \ne 0\right).$$

We show that if a non-decreasing function $\sigma(t)$ is the spectral distribution function of a positive self-adjoint boundary value problem

$$\begin{cases} A_\theta y = -y'' + q(x)y \\ y'(a) = \theta y(a) \end{cases}$$

and satisfies conditions

$$\int_0^\infty d\sigma(t) = \infty, \quad \int_0^\infty \frac{d\sigma(t)}{t + t^2} < \infty,$$

then for every real α an inverse Stieltjes-like function

$$V(z) = \alpha + \int_0^\infty \left(\frac{1}{t - z} - \frac{1}{t}\right) d\sigma(t)$$

can be realized in the unique way as $V(z) = V_\Theta(z) = i[W_\Theta(z) + I]^{-1}[W_\Theta(z) - I]$, where $W_\Theta(z)$ is the transfer function of a rigged canonical system Θ containing some Schrödinger operator T_h. In particular, it is shown that for every $\alpha \le 0$ an inverse Stieltjes function $V(z)$ with integral representation above can be realized by a system Θ whose main operator $\mathbb{A}$ is a $(*)$-extension of a Schrödinger operator T_h and satisfies (2.7).

In addition to the general realization results, Section 5 provides the reader with formulas that allow to find the exact value of a non-real parameter h in the definition of T_h of the realizing system Θ. A somewhat similar study is presented in Section 6 to describe the real parameter μ that appears in the construction of the elements of the realizing system. An elaborate investigation of these formulas shows the dynamics of the restored parameters h and μ in terms of a changing free term α in the integral representation of $V(z)$ above. It will be shown and graphically presented that the parametric equations for the restored parameter h represent different circles whose centers and radii are completely determined by the function $V(z)$. Similarly, the behavior of the restored parameter μ are described by straight lines.

2. Some preliminaries

For a pair of Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ we denote by $[\mathcal{H}_1, \mathcal{H}_2]$ the set of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. Let A be a closed, densely defined, symmetric operator in a Hilbert space $\mathcal{H}$ with inner product $(f, g), f, g \in \mathcal{H}$. Consider the rigged Hilbert space

$$\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-,$$

where $\mathcal{H}_+ = D(A^*)$ and

$$(f, g)_+ = (f, g) + (A^*f, A^*g), \quad f, g \in D(A^*).$$

Note that identifying the space conjugate to $\mathcal{H}_\pm$ with $\mathcal{H}_\mp$, we get that if $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ then $\mathbb{A}^* \in [\mathcal{H}_+, \mathcal{H}_-]$.

Definition 2.1. An operator $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ is called a self-adjoint bi-extension of a symmetric operator A if $\mathbb{A} = \mathbb{A}^*$, $\mathbb{A} \supset A$, and the operator

$$\widehat{A}f = \mathbb{A}f, \; f \in D(\widehat{A}) = \{f \in \mathcal{H}_+ : \mathbb{A}f \in \mathcal{H}\}$$

is self-adjoint in $\mathcal{H}$.

The operator $\widehat{A}$ in the above definition is called a *quasi-kernel* of a self-adjoint bi-extension $\mathbb{A}$ (see [30]) .

Definition 2.2. An operator $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ is called a $(*)$-extension (or correct bi-extension) of an operator T (with non-empty set $\rho(T)$ of regular points) if

$$\mathbb{A} \supset T \supset A, \mathbb{A}^* \supset T^* \supset A$$

and the operator $\mathbb{A}_R = \frac{1}{2}(\mathbb{A} + \mathbb{A}^*)$ is a self-adjoint bi-extension of an operator A.

The existence, description, and analog of von Neumann's formulas for self-adjoint bi-extensions and $(*)$-extensions were discussed in [30] (see also [4], [5], [11]). For instance, if Φ is an isometric operator from the defect subspace $\mathfrak{N}_i$ of the symmetric operator A onto the defect subspace $\mathfrak{N}_{-i}$, then the formulas below establish a one-to one correspondence between $(*)$-extensions of an operator T and Φ

$$\mathbb{A}f = A^*f + iR(\Phi - I)x, \; \mathbb{A}^*f = A^*f + iR(\Phi - I)y, \tag{2.1}$$

where $x, y \in \mathfrak{N}_i$ are uniquely determined from the conditions

$$f - (\Phi + I)x \in D(T), \; f - (\Phi + I)y \in D(T^*)$$

and R is the Riesz-Berezanskii operator of the triplet $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ that maps $\mathcal{H}_+$ isometrically onto $\mathcal{H}_-$ (see [30]). If the symmetric operator A has deficiency indices (n, n), then formulas (2.1) can be rewritten in the following form

$$\mathbb{A}f = A^*f + \sum_{k=1}^{n} \Delta_k(f) V_k, \quad \mathbb{A}^*f = A^*f + \sum_{k=1}^{n} \delta_k(f) V_k, \tag{2.2}$$

where $\{V_j\}_1^n \in \mathcal{H}_-$ is a basis in the subspace $R(\Phi - I)\mathfrak{N}_i$, and $\{\Delta_k\}_1^n$, $\{\delta_k\}_1^n$, are bounded linear functionals on $\mathcal{H}_+$ with the properties

$$\Delta_k(f) = 0, \ \ \forall f \in D(T), \quad \delta_k(f) = 0, \ \ \forall f \in D(T^*). \tag{2.3}$$

Let $\mathcal{H} = L_2[a, +\infty)$ and $l(y) = -y'' + q(x)y$ where q is a real locally summable function. Suppose that the symmetric operator

$$\begin{cases} Ay = -y'' + q(x)y \\ y(a) = y'(a) = 0 \end{cases} \tag{2.4}$$

has deficiency indices (1,1). Let D^* be the set of functions locally absolutely continuous together with their first derivatives such that $l(y) \in L_2[a, +\infty)$. Consider $\mathcal{H}_+ = D(A^*) = D^*$ with the scalar product

$$(y, z)_+ = \int_a^\infty \left(y(x)\overline{z(x)} + l(y)\overline{l(z)} \right) dx, \quad y, \ z \in D^*.$$

Let

$$\mathcal{H}_+ \subset L_2[a, +\infty) \subset \mathcal{H}_-$$

be the corresponding triplet of Hilbert spaces. Consider operators

$$\begin{cases} T_h y = l(y) = -y'' + q(x)y \\ hy(a) - y'(a) = 0 \end{cases}, \quad \begin{cases} T_h^* y = l(y) = -y'' + q(x)y \\ \overline{h}y(a) - y'(a) = 0 \end{cases}, \tag{2.5}$$

$$\begin{cases} \widehat{A}y = l(y) = -y'' + q(x)y \\ \mu y(a) - y'(a) = 0 \end{cases}, \quad \operatorname{Im} \mu = 0.$$

It is well known [1] that $\widehat{A} = \widehat{A^*}$. The following theorem was proved in [6].

Theorem 2.3. *The set of all $(*)$-extensions of a non-self-adjoint Schrödinger operator T_h of the form* (2.5) *in $L_2[a, +\infty)$ can be represented in the form*

$$\begin{aligned} \mathbb{A}y &= -y'' + q(x)y - \frac{1}{\mu - h}\,[y'(a) - hy(a)]\,[\mu\delta(x-a) + \delta'(x-a)], \\ \mathbb{A}^*y &= -y'' + q(x)y - \frac{1}{\mu - \overline{h}}\,[y'(a) - \overline{h}y(a)]\,[\mu\delta(x-a) + \delta'(x-a)]. \end{aligned} \tag{2.6}$$

In addition, the formulas (2.6) *establish a one-to-one correspondence between the set of all $(*)$-extensions of a Schrödinger operator T_h of the form* (2.5) *and all real numbers $\mu \in [-\infty, +\infty]$.*

Definition 2.4. An operator T with the domain $D(T)$ and $\rho(T) \neq \emptyset$ acting on a Hilbert space $\mathcal{H}$ is called *accretive* if

$$\operatorname{Re}(Tf, f) \geq 0, \ \ \forall f \in D(T).$$

Definition 2.5. An accretive operator T is called [22] *α-sectorial* if there exists a value of $\alpha \in (0, \pi/2)$ such that

$$\cot\alpha\,|\mathrm{Im}\,(Tf, f)| \le \mathrm{Re}\,(Tf, f), \qquad f \in \mathcal{D}(T).$$

An accretive operator is called *extremal accretive* if it is not α-sectorial for any $\alpha \in (0, \pi/2)$.

Definition 2.6. A $(*)$-extensions $\mathbb{A}$ in Definition 2.2 is called *accumulative* if

$$(\mathbb{A}_R f, f) \le (A^* f, f) + (f, A^* f), \quad f \in \mathcal{H}_+. \tag{2.7}$$

Consider the symmetric operator A of the form (2.4) with defect indices (1,1), generated by the differential operation $l(y) = -y'' + q(x)y$. Let $\varphi_k(x, \lambda)\,(k = 1, 2)$ be the solutions of the following Cauchy problems:

$$\begin{cases} l(\varphi_1) = \lambda\varphi_1 \\ \varphi_1(a, \lambda) = 0 \\ \varphi_1'(a, \lambda) = 1 \end{cases}, \begin{cases} l(\varphi_2) = \lambda\varphi_2 \\ \varphi_2(a, \lambda) = -1 \\ \varphi_2'(a, \lambda) = 0 \end{cases}.$$

It is well known [1] that there exists a function $m_\infty(\lambda)$ (called the Weyl-Titchmarsh function) for which

$$\varphi(x, \lambda) = \varphi_2(x, \lambda) + m_\infty(\lambda)\varphi_1(x, \lambda)$$

belongs to $L_2[a, +\infty)$.

Suppose that the symmetric operator A of the form (2.4) with deficiency indices (1,1) is nonnegative, i.e., $(Af, f) \ge 0$ for all $f \in D(A)$). It was shown in [28] that the Schrödinger operator T_h of the form (2.5) is accretive if and only if

$$\mathrm{Re}\,h \ge -m_\infty(-0). \tag{2.8}$$

For real h such that $h \ge -m_\infty(-0)$ we get a description of all nonnegative self-adjoint extensions of an operator A. For $h = -m_\infty(-0)$ the corresponding operator

$$\begin{cases} A_K\, y = -y'' + q(x)y \\ y'(a) + m_\infty(-0)y(a) = 0 \end{cases} \tag{2.9}$$

is the Kreĭn-von Neumann extension of A and for $h = +\infty$ the corresponding operator

$$\begin{cases} A_F\, y = -y'' + q(x)y \\ y(a) = 0 \end{cases} \tag{2.10}$$

is the Friedrichs extension of A (see [28], [6]).

3. Rigged canonical systems with Schrödinger operator

Let $\mathbb{A}$ be $(*)$-extension of an operator T, i.e.,

$$\mathbb{A} \supset T \supset A, \quad \mathbb{A}^* \supset T^* \supset A$$

where A is a symmetric operator with deficiency indices (n, n) and $D(A) = D(T) \cap D(T^*)$. In what follows we will only consider the case when the symmetric operator A has dense domain, i.e., $\overline{\mathcal{D}(A)} = \mathcal{H}$.

Definition 3.1. A system of equations

$$\begin{cases} (\mathbb{A} - zI)x = KJ\varphi_- \\ \varphi_+ = \varphi_- - 2iK^*x \end{cases},$$

or an array

$$\Theta = \begin{pmatrix} \mathbb{A} & K & J \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & E \end{pmatrix} \tag{3.1}$$

is called a *rigged canonical system of the Livsic type* if:

1) E is a finite-dimensional Hilbert space with scalar product $(\cdot,\cdot)_E$ and the operator J in this space satisfies the conditions $J = J^* = J^{-1}$,
2) $K \in [E, \mathcal{H}_-]$, $\ker K = \{0\}$,
3) $\operatorname{Im} \mathbb{A} = KJK^*$, where $K^* \in [\mathcal{H}_+, E]$ is the adjoint of K.

In the definition above $\varphi_- \in E$ stands for an input vector, $\varphi_+ \in E$ is an output vector, and x is a state space vector in $\mathcal{H}$. An operator $\mathbb{A}$ is called a *main operator* of the system Θ, J is a *direction operator*, and K is a *channel operator*. A system Θ of the form (3.1) is called an *accretive system* [14] if its main operator $\mathbb{A}$ is accretive and *accumulative* if its main operator $\mathbb{A}$ is accumulative, i.e., satisfies (2.7).

An operator-valued function

$$W_\Theta(\lambda) = I - 2iK^*(\mathbb{A} - \lambda I)^{-1}KJ \tag{3.2}$$

defined on the set $\rho(T)$ of regular points of an operator T is called the *transfer function* (*characteristic function*) of the system Θ, i.e., $\varphi_+ = W_\Theta(\lambda)\varphi_-$. It is known [28], [30] that any $(*)$-extension $\mathbb{A}$ of an operator T ($A^* \supset T \supset A$), where A is a symmetric operator with deficiency indices (n,n) ($n < \infty$), $D(A) = D(T) \cap D(T^*)$, can be included as a main operator of some rigged canonical system with $\dim E < \infty$ and invertible channel operator K.

It was also established [28], [30] that

$$V_\Theta(\lambda) = K^*(\operatorname{Re} \mathbb{A} - \lambda I)^{-1}K \tag{3.3}$$

is a Herglotz-Nevanlinna operator-valued function acting on a Hilbert space E, satisfying the following relation for $\lambda \in \rho(T)$, $\operatorname{Im} \lambda \neq 0$

$$V_\Theta(\lambda) = i[W_\Theta(\lambda) - I][W_\Theta(\lambda) + I]^{-1}J. \tag{3.4}$$

Alternatively,

$$\begin{aligned} W_\Theta(\lambda) &= (I + iV_\Theta(\lambda)J)^{-1}(I - iV_\Theta(\lambda)J) \\ &= (I - iV_\Theta(\lambda)J)(I + iV_\Theta(\lambda)J)^{-1}. \end{aligned} \tag{3.5}$$

Let us recall (see [30], [6]) that a symmetric operator with dense domain $\mathcal{D}(A)$ is called *prime* if there is no reducing, nontrivial invariant subspace on which A induces a self-adjoint operator. It was established in [29] that a symmetric operator A is prime if and only if

$$\underset{\lambda \neq \bar{\lambda}}{c.l.s.}\, \mathfrak{N}_\lambda = \mathcal{H}. \tag{3.6}$$

We call a rigged canonical system of the form (3.1) *prime* if

$$\underset{\lambda\neq\bar\lambda,\,\lambda\in\rho(T)}{c.l.s.}\ \mathfrak{N}_\lambda=\mathcal{H}.$$

One easily verifies that if system Θ is prime, then a symmetric operator A of the system is prime as well.

The following theorem [6], [14] and corollary [14] establish the connection between two rigged canonical systems with equal transfer functions.

Theorem 3.2. *Let* $\Theta_1=\begin{pmatrix}\mathbb{A}_1 & K_1 & J\\ \mathcal{H}_{+1}\subset\mathcal{H}_1\subset\mathcal{H}_{-1} & & E\end{pmatrix}$ *and* $\Theta_2=\begin{pmatrix}\mathbb{A}_2 & K_2 & J\\ \mathcal{H}_{+2}\subset\mathcal{H}_2\subset\mathcal{H}_{-2} & & E\end{pmatrix}$ *be two prime rigged canonical systems of the Livsic type with*

$$\begin{aligned}\mathbb{A}_1\supset T_1\supset A_1,\quad \mathbb{A}_1^*\supset T_1^*\supset A_1,\\ \mathbb{A}_2\supset T_2\supset A_2,\quad \mathbb{A}_2^*\supset T_2^*\supset A_2,\end{aligned}\tag{3.7}$$

and such that A_1 and A_2 have finite and equal defect indices.

If

$$W_{\Theta_1}(\lambda)=W_{\Theta_2}(\lambda),\tag{3.8}$$

then there exists an isometric operator U from $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that $U_+=U|_{\mathcal{H}_{+1}}$ is an isometry[1] *from $\mathcal{H}_{+1}$ onto $\mathcal{H}_{+2}$, $U_-^*=U_+^*$ is an isometry from $\mathcal{H}_{-1}$ onto $\mathcal{H}_{-2}$, and*

$$UT_1=T_2U,\quad \mathbb{A}_2=U_-\mathbb{A}_1U_+^{-1},\quad U_-K_1=K_2.\tag{3.9}$$

Corollary 3.3. *Let Θ_1 and Θ_2 be the two prime systems from the statement of Theorem* 3.2. *Then the mapping U described in the conclusion of the theorem is unique.*

Now we shall construct a rigged canonical system based on a non-self-adjoint Schrödinger operator. One can easily check that the $(*)$-extension

$$\mathbb{A}y=-y''+q(x)y-\frac{1}{\mu-h}\,[y'(a)-hy(a)]\,[\mu\delta(x-a)+\delta'(x-a)],\ \ \operatorname{Im}h>0$$

of the non-self-adjoint Schrödinger operator T_h of the form (2.5) satisfies the condition

$$\operatorname{Im}\mathbb{A}=\frac{\mathbb{A}-\mathbb{A}^*}{2i}=(.,g)g,\tag{3.10}$$

where

$$g=\frac{(\operatorname{Im}h)^{\frac12}}{|\mu-h|}\,[\mu\delta(x-a)+\delta'(x-a)]\tag{3.11}$$

[1]It was shown in [6] that the operator U_+ defined this way is an isometry from $\mathcal{H}_{+1}$ onto $\mathcal{H}_{+2}$. It is also shown there that the isometric operator $U^*:\mathcal{H}_{+2}\to\mathcal{H}_{+1}$ uniquely defines operator $U_-=(U^*)^*:\mathcal{H}_{-1}\to\mathcal{H}_{-2}$.

and $\delta(x-a), \delta'(x-a)$ are the delta-function and its derivative at the point a. Moreover,

$$(y,g) = \frac{(\operatorname{Im} h)^{\frac{1}{2}}}{|\mu - h|}\,[\mu y(a) - y'(a)], \tag{3.12}$$

where

$$y \in \mathcal{H}_+, g \in \mathcal{H}_-, \mathcal{H}_+ \subset L_2(a,+\infty) \subset \mathcal{H}_-$$

and the triplet of Hilbert spaces is as discussed in Theorem 2.3. Let $E = \mathbb{C}$, $Kc = cg$ ($c \in \mathbb{C}$). It is clear that

$$K^* y = (y,g), \quad y \in \mathcal{H}_+ \tag{3.13}$$

and $\operatorname{Im} \mathbb{A} = KK^*$. Therefore, the array

$$\Theta = \begin{pmatrix} \mathbb{A} & K & 1 \\ \mathcal{H}_+ \subset L_2[a,+\infty) \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix} \tag{3.14}$$

is a rigged canonical system with the main operator $\mathbb{A}$ of the form (2.6), the direction operator $J = 1$ and the channel operator K of the form (3.13). Our next logical step is finding the transfer function of (3.14). It was shown in [6] that

$$W_\Theta(\lambda) = \frac{\mu - h}{\mu - \overline{h}}\,\frac{m_\infty(\lambda) + \overline{h}}{m_\infty(\lambda) + h}, \tag{3.15}$$

and

$$V_\Theta(\lambda) = \frac{(m_\infty(\lambda) + \mu)\operatorname{Im} h}{(\mu - \operatorname{Re} h)\, m_\infty(\lambda) + \mu \operatorname{Re} h - |h|^2}. \tag{3.16}$$

4. Realization of inverse Stieltjes functions

Let E be a finite-dimensional Hilbert space. The scalar versions of the definitions below can be found in [21]. We recall (see [14], [21]) that an operator-valued Herglotz-Nevanlinna function $V(z)$ is Stieltjes if it is holomorphic in $\operatorname{Ext}[0,+\infty)$ and

$$\frac{\operatorname{Im}[zV(z)]}{\operatorname{Im} z} \geq 0.$$

Definition 4.1. We will call an operator-valued Herglotz-Nevanlinna function $V(z) \in [E,E]$ by an *inverse Stieltjes* if $V(z)$ admits the following integral representation

$$V(z) = \alpha + \beta \cdot z + \int_0^\infty \left(\frac{1}{t-z} - \frac{1}{t}\right) dG(t), \tag{4.1}$$

where $\alpha \leq 0$, $\beta \geq 0$, and $G(t)$ is a non-decreasing on $[0,+\infty)$ operator-valued function such that

$$\int_0^\infty \frac{(dG(t)e,e)}{t+t^2} < \infty, \quad \forall e \in E.$$

Alternatively (see [21]) an operator-valued function $V(z)$ is inverse Stieltjes if it is holomorphic in $\mathrm{Ext}[0,+\infty)$ and $V(z) \le 0$ in $(-\infty, 0)$. It is known [21] that a function $V(z) \neq 0$ is an inverse Stieltjes function iff the function $-(V(z))^{-1}$ is Stieltjes.

The following definition was given in [13] and provides the description of all realizable inverse Stieltjes operator-valued functions.

Definition 4.2. An operator-valued inverse Stieltjes function $V(z) \in [E, E]$ is said to be a member of the class $S^{-1}(R)$ if in the representation (4.1) we have

$$\text{i)} \qquad \beta = 0,$$

$$\text{ii)} \qquad \alpha e = \int_0^\infty \frac{1}{t} dG(t)e = 0,$$

for all $e \in E$ with

$$\int_0^\infty (dG(t)e, e)_E < \infty. \tag{4.2}$$

In what follows we will, however, be mostly interested in the following subclass of $S^{-1}(R)$ that was also introduced in [13].

Definition 4.3. An operator-valued inverse Stieltjes function $V(z) \in S^{-1}(R)$ is a member of the class $S_0^{-1}(R)$ if

$$\int_0^\infty (dG(t)e, e)_E = \infty, \tag{4.3}$$

for all $e \in E$, $e \neq 0$.

It is not hard to see that $S_0^{-1}(R)$ is the analogue of the class $N_0(R)$ introduced in [12] and of the class $S_0(R)$ discussed in [14].

The following statement [13] is the direct realization theorem for the functions of the class $S_0^{-1}(R)$.

Theorem 4.4. *Let Θ be an accumulative system of the form* (3.1). *Then the operator-function $V_\Theta(z)$ of the form* (3.3), (3.4) *belongs to the class $S_0^{-1}(R)$.*

The inverse realization theorem can be stated and proved (see [13]) for the class $S_0^{-1}(R)$ as follows.

Theorem 4.5. *Let a operator-valued function $V(z)$ belong to the class $S_0^{-1}(R)$. Then $V(z)$ admits a realization by an accumulative prime system Θ of the form* (3.1) *with $J = I$.*

Proof. It was shown in [13] that any member of the class $S_0^{-1}(R)$ is realizable by an accumulative system Θ of the form (3.1) with $J = I$. Thus all we actually have to show is that the model system Θ that was constructed in [13] is prime.

As it was also shown in [11], [12], and [13], the symmetric operator A of the model system Θ is prime and positive, and hence (3.6) takes place. We are going to show that in this case the system Θ is also prime, i.e.,

$$\underset{\lambda \neq \bar{\lambda},\, \lambda \in \rho(T)}{c.l.s.} \mathfrak{N}_\lambda = \mathcal{H}. \tag{4.4}$$

Consider the operator $U_{\lambda_0\lambda} = (\tilde{A} - \lambda_0 I)(\tilde{A} - \lambda I)^{-1}$, where $\tilde{A}$ is an arbitrary self-adjoint extension of A. By a simple check one confirms that $U_{\lambda_0\lambda}\mathfrak{N}_{\lambda_0} = \mathfrak{N}_\lambda$. To prove (4.4) we assume that there is a function $f \in \mathcal{H}$ such that

$$f \perp \underset{\lambda\neq\bar{\lambda},\,\lambda\in\rho(T)}{c.l.s.} \mathfrak{N}_\lambda.$$

Then $(f, U_{\lambda_0\lambda}g) = 0$ for all $g \in \mathfrak{N}_{\lambda_0}$ and all $\lambda \in \rho(T)$. But since the system Θ is accumulative, it follows that there are regular points of T in the upper and lower half-planes. This leads to a conclusion that the function $\phi(\lambda) = (f, U_{\lambda_0\lambda}g) \equiv 0$ for all $\lambda \neq \bar{\lambda}$. Combining this with (3.6) we conclude that $f = 0$ and thus (4.4) holds. $\square$

5. Restoring a non-self-adjoint Schrödinger operator T_h

In this section we are going to use the realization technique and results developed for inverse Stieltjes functions in section 4 to obtain the solution of inverse spectral problem for Schrödinger operator of the form (2.5) in $L_2[a, +\infty)$ with non-self-adjoint boundary conditions

$$\begin{cases} T_h y = -y'' + q(x)y \\ y'(a) = hy(a) \end{cases}, \quad \left(q(x) = \overline{q(x)},\ \operatorname{Im} h \neq 0\right). \tag{5.1}$$

Following the framework of [14] we let $\mathcal{H} = L_2[a, +\infty)$ and $l(y) = -y'' + q(x)y$ where q is a real locally summable function. We consider a symmetric operator with defect indices $(1, 1)$

$$\begin{cases} \tilde{B}y = -y'' + q(x)y \\ y'(a) = y(a) = 0 \end{cases} \tag{5.2}$$

together with its positive self-adjoint extension of the form

$$\begin{cases} \tilde{B}_\theta y = -y'' + q(x)y \\ y'(a) = \theta y(a) \end{cases} \tag{5.3}$$

defined in $\mathcal{H} = L_2[a, +\infty)$. A non-decreasing function $\sigma(\lambda)$ defined on $[0, +\infty)$ is called the *distribution function* (see [25]) *of an operator pair* $\tilde{B}_\theta$, $\tilde{B}$, where $\tilde{B}_\theta$ of the form (5.3) is a self-adjoint extension of symmetric operator $\tilde{B}$ of the form (5.2), and if the formulas

$$\begin{aligned} \varphi(\lambda) &= Uf(x), \\ f(x) &= U^{-1}\varphi(\lambda), \end{aligned} \tag{5.4}$$

establish one-to-one isometric correspondence U between

$$L_2^\sigma[0, +\infty) \quad \text{and} \quad L_2[a, +\infty).$$

Moreover, this correspondence is such that the operator $\tilde{B}_\theta$ is unitarily equivalent to the operator

$$\Lambda_\sigma\varphi(\lambda) = \lambda\varphi(\lambda), \quad (\varphi(\lambda) \in L_2^\sigma[0, +\infty)) \tag{5.5}$$

in $L_2^\sigma[0,+\infty)$ while symmetric operator $\tilde{B}$ in (5.2) is unitarily equivalent to the symmetric operator

$$\Lambda_\sigma^0\varphi(\lambda)=\lambda\varphi(\lambda),\quad D(\Lambda_\sigma^0)=\left\{\varphi(\lambda)\in L_2^\sigma[0,+\infty):\ \int_0^{+\infty}\varphi(\lambda)d\sigma(\lambda)=0\right\}. \tag{5.6}$$

Definition 5.1. A scalar Herglotz-Nevanlinna function $V(z)$ is called an *inverse Stieltjes-like function* if it has an integral representation

$$V(z)=\alpha+\int_0^\infty\left(\frac{1}{t-z}-\frac{1}{t}\right)d\tau(t),\quad \int_0^\infty\frac{d\tau(t)}{t+t^2}<\infty \tag{5.7}$$

similar to (4.1) but with an arbitrary (not necessarily non-positive) constant α.

We are going to introduce a new class of realizable scalar inverse Stieltjes-like functions whose structure is similar to that of $S_0^{-1}(R)$ of Section 4.

Definition 5.2. An inverse Stieltjes-like function $V(z)$ is said to be a member of the *class* $SL_0^{-1}(R)$ if it admits an integral representation

$$V(z)=\alpha+\int_0^\infty\left(\frac{1}{t-z}-\frac{1}{t}\right)d\tau(t), \tag{5.8}$$

where non-decreasing function $\tau(t)$ satisfies the following conditions

$$\int_0^\infty d\tau(t)=\infty,\quad \int_0^\infty\frac{d\tau(t)}{t+t^2}<\infty. \tag{5.9}$$

Consider the following subclasses of $SL_0^{-1}(R)$.

Definition 5.3. A function $V(z)\in SL_0^{-1}(R)$ belongs to the *class* $SL_0^{-1}(R,K)$ if

$$\int_0^\infty\frac{d\tau(t)}{t}=\infty. \tag{5.10}$$

Definition 5.4. A function $V(z)\in SL_0^{-1}(R)$ belongs to the *class* $SL_{01}^{-1}(R,K)$ if

$$\int_0^\infty\frac{d\tau(t)}{t}<\infty. \tag{5.11}$$

The following theorem describes the realization of the class $SL_0^{-1}(R)$.

Theorem 5.5. *Let $V(z)\in SL_0^{-1}(R)$. Then it can be realized by a prime system Θ of the form* (3.1).

Proof. We start by applying the general realization theorems from [11] and [13] to a Herglotz-Nevanlinna function $V(z)$ and obtain a rigged canonical system of the Livsic type

$$\Theta_\Lambda=\begin{pmatrix}\mathbf{\Lambda} & K^\tau & 1\\ \mathcal{H}_+^\tau\subset L_2^\tau[0,+\infty)\subset\mathcal{H}_-^\tau & & \mathbb{C}\end{pmatrix}, \tag{5.12}$$

such that $V(z) = V_{\Theta_\Lambda}(z)$. Following the steps for construction of the model system described in [11] and [13], we note that

$$\mathbf{\Lambda} = \operatorname{Re}\mathbf{\Lambda} + iK^\tau(K^\tau)^*$$

is a correct $(*)$-extension of an operator T^τ such that $\mathbf{\Lambda} \supset T^\tau \supset \Lambda_\tau^0$ where Λ_τ^0 is defined in (5.6). The real part $\operatorname{Re}\mathbf{\Lambda}$ is a self-adjoint bi-extension of Λ_τ^0 that has a quasi-kernel Λ_τ of the form (5.5). It was also shown in [13] that the operator $\mathbf{\Lambda}$ possess the accumulative property (2.7). The operator K^τ in the above system (see [11], [13]) is defined by

$$K^\tau c = c \cdot \alpha, \quad (K^\tau)^* x = (x, \alpha) \quad c \in \mathbb{C},\ \alpha \in \mathcal{H}_-^\tau,\ x(t) \in \mathcal{H}_+^\tau.$$

In addition we can observe that the function $\eta(\lambda) \equiv 1$ belongs to $\mathcal{H}_-^\tau$. To confirm this we need to show that $(x, 1)$ defines a continuous linear functional for every $x \in \mathcal{H}_+^\tau$. It was shown in [11], [12] that

$$\mathcal{H}_+^\tau = \mathcal{D}(\Lambda_\tau^0) \dotplus \left\{\frac{c_1}{1+t^2}\right\} \dotplus \left\{\frac{c_2 t}{1+t^2}\right\}, \quad c_1, c_2 \in \mathbb{C}. \tag{5.13}$$

Consequently, every vector $x \in \mathcal{H}_+^\tau$ has three components $x = x_1 + x_2 + x_3$ according to the decomposition (5.13) above. Obviously, $(x_1, 1)$ and $(x_2, 1)$ yield convergent integrals while $(x_3, 1)$ boils down to

$$\int_0^\infty \frac{t}{1+t^2}\, d\tau(t).$$

The convergence of the latter is guaranteed by the definition of inverse Stieltjes-like function. The state space of the system Θ_Λ is $\mathcal{H}_+^\tau \subset L_2^\tau[0, +\infty) \subset \mathcal{H}_-^\tau$, where $\mathcal{H}_+^\tau = \mathcal{D}\big((\Lambda_\tau^0)^*\big)$.

We can also show that the system Θ_Λ is a prime system. In order to do so we need to show that

$$\underset{\lambda \neq \bar\lambda,\ \lambda \in \rho(T^\tau)}{c.l.s.} \mathfrak{N}_\lambda = L_2^\tau[0, +\infty), \tag{5.14}$$

where $\mathfrak{N}_\lambda$ are defect subspaces of the symmetric operator Λ_τ^0. It is known (see [11], [13]) that Λ_τ^0 is a non-negative prime operator. Hence we can follow the reasoning of the proof of theorem 4.5 and only confirm that operator T^τ has regular points in the upper and lower half-planes. To see this we first note that non-negative operator Λ_τ^0 has no kernel spectrum [1] on the left real half-axis. Then we apply Theorem 1 of [1] (see page 149 of vol. 2 of [1]) that gives the complete description of the spectrum of T^τ. This theorem implies that there are regular points of T^τ on the left real half-axis. Since $\rho(T^\tau)$ is an open set we confirm the presence of non-real regular points of T^τ in both half-planes. Thus (5.14) holds and Θ_Λ is a prime system.

In order to complete the proof of the theorem we merely set

$$\mathbb{A} = \mathbf{\Lambda} = \operatorname{Re}\mathbf{\Lambda} + iK^\tau(K^\tau)^* \quad \text{and} \quad K = K^\tau. \qquad \square$$

At this point we are ready to state and prove the main realization result of this paper.

Theorem 5.6. *Let $V(z) \in SL_0^{-1}(R)$ and the function $\tau(t)$ be the distribution function of an operator pair $\tilde{B}_\theta$ of the form* (5.2) *and $\tilde{B}$ of the form* (5.3). *Then there exist unique Schrödinger operator T_h $(\operatorname{Im} h > 0)$ of the form* (5.1), *operator $\mathbb{A}$ given by* (2.6), *operator K as in* (3.13), *and the rigged canonical system of the Livsic type*

$$\Theta = \begin{pmatrix} \mathbb{A} & K & 1 \\ \mathcal{H}_+ \subset L_2[a,+\infty) \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix}, \tag{5.15}$$

of the form (3.14) *so that $V(z)$ is realized by Θ, i.e., $V(z) = V_\Theta(z)$.*

Proof. Since $\tau(t)$ is the distribution function of the positive self-adjoint operator, then (see [25]) we can completely restore the operator $\tilde{B}_\theta$ of the form (5.3) as well as a symmetric operator $\tilde{B}$ of the form (5.2). It follows from the definition of the distribution function above that there is operator U defined in (5.4) establishing one-to-one isometric correspondence between $L_2^\tau[0,+\infty)$ and $L_2[a,+\infty)$ while providing for the unitary equivalence between the operator $\tilde{B}_\theta$ and operator of multiplication by independent variable Λ_τ of the form (5.5).

Let us consider the system Θ_Λ of the form (5.12) constructed in the proof of Theorem 5.5. Applying Theorem 3.2 on unitary equivalence to the isometry U defined in (5.4) we obtain a triplet of isometric operators U_+, U, and U_-, where

$$U_+ = U\big|_{\mathcal{H}_+^\tau}, \quad U_-^* = U_+^*.$$

This triplet of isometric operators will map the rigged Hilbert space of Θ_Λ, that is $\mathcal{H}_+^\tau \subset L_2^\tau[0,+\infty) \subset \mathcal{H}_-^\tau$, into another rigged Hilbert space $\mathcal{H}_+ \subset L_2^\tau[a,+\infty) \subset \mathcal{H}_-$. Moreover, U_+ is an isometry from $\mathcal{H}_+^\tau = \mathcal{D}(\Lambda_\tau^{0*})$ onto $\mathcal{H}_+ = \mathcal{D}(\tilde{B}^*)$, and $U_-^* = U_+^*$ is an isometry from $\mathcal{H}_+^\tau$ onto $\mathcal{H}_-$. This is true since the operator U provides the unitary equivalence between the symmetric operators $\tilde{B}$ and Λ_τ^0.

Now we construct a system

$$\Theta = \begin{pmatrix} \mathbb{A} & K & 1 \\ \mathcal{H}_+ \subset L_2[a,+\infty) \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix}$$

where $K = U_- K^\tau$ and $\mathbb{A} = U_- \mathbf{\Lambda} U_+^{-1}$ is a correct $(*)$-extension of operator $T = UT^\tau U^{-1}$ such that $\mathbb{A} \supset T \supset \tilde{B}$. The real part $\operatorname{Re}\mathbb{A}$ contains the quasi-kernel $\tilde{B}_\theta$. This construction of $\mathbb{A}$ is unique due to the theorem on the uniqueness of a $(*)$-extension for a given quasi-kernel (see [30]). On the other hand, all $(*)$-extensions based on a pair $\tilde{B}$, $\tilde{B}_\theta$ must take form (2.6) for some values of parameters h and μ. Consequently, our function $V(z)$ is realized by the system Θ of the form (5.15) and

$$V(z) = V_{\Theta_\Lambda}(z) = V_\Theta(z). \qquad \square$$

The theorem below gives the criteria for the operator T_h of the realizing system to be accretive.

Theorem 5.7. *Let $V(z) \in SL_0^{-1}(R)$ satisfy the conditions of Theorem 5.5. Then the operator T_h in the conclusion of the Theorem 5.5 is accretive if and only if*

$$\alpha^2 - \alpha \int_0^\infty \frac{d\tau(t)}{t} + 1 \geq 0. \tag{5.16}$$

The operator T_h is ϕ-sectorial for some $\phi \in (0, \pi/2)$ if and only if the inequality (5.16) is strict. In this case the exact value of angle ϕ can be calculated by the formula

$$\tan\phi = \frac{\int_0^\infty \frac{d\tau(t)}{t}}{\alpha^2 - \alpha\int_0^\infty \frac{d\tau(t)}{t} + 1}. \tag{5.17}$$

Proof. It was shown in [29] that for the system Θ in (5.15) described in the previous theorem the operator T_h is accretive if and only if the function

$$\begin{aligned} V_h(z) &= -i[W_\Theta^{-1}(-1)W_\Theta(z) + I]^{-1}[W_\Theta^{-1}(-1)W_\Theta(z) - I] \\ &= -i\frac{1 - [(m_\infty(z) + \bar{h})/(m_\infty(z) + h)][(m_\infty(-1) + h)/(m_\infty(-1) + \bar{h})]}{1 + [(m_\infty(z) + \bar{h})/(m_\infty(z) + h)][(m_\infty(-1) + h)/(m_\infty(-1) + \bar{h})]}, \end{aligned} \tag{5.18}$$

is holomorphic in $\mathrm{Ext}[0, +\infty)$ and satisfies the following inequality

$$1 + V_h(0)\,V_h(-\infty) \geq 0. \tag{5.19}$$

Here $W_\Theta(z)$ is the transfer function of (5.15). It is also shown in [29] that the operator T_h is α-sectorial for some $\alpha \in (0, \pi/2)$ if and only if the inequality (5.19) is strict while the exact value of angle α can be calculated by the formula

$$\cot\alpha = \frac{1 + V_h(0)\,V_h(-\infty)}{|V_h(-\infty) - V_h(0)|}. \tag{5.20}$$

According to Theorem 5.5 and equation (3.5)

$$W_\Theta(z) = (I - iV(z)J)(I + iV(z)J)^{-1}.$$

By direct calculations one obtains

$$W_\Theta(-1) = \frac{1 - i\left[\alpha - \int_0^\infty \frac{d\tau(t)}{t+t^2}\right]}{1 + i\left[\alpha - \int_0^\infty \frac{d\tau(t)}{t+t^2}\right]},\ W_\Theta^{-1}(-1) = \frac{1 + i\left[\alpha - \int_0^\infty \frac{d\tau(t)}{t+t^2}\right]}{1 - i\left[\alpha - \int_0^\infty \frac{d\tau(t)}{t+t^2}\right]}. \tag{5.21}$$

Using the following notations

$$c = \alpha - \int_0^\infty \frac{d\tau(t)}{t+t^2} \quad \text{and} \quad d = \alpha - \int_0^\infty \frac{d\tau(t)}{t},$$

and performing straightforward calculations we obtain

$$W_\Theta(-1) = \frac{1 - i\,c}{1 + i\,c}, \quad W_\Theta(-\infty) = \frac{1 - i\,d}{1 + i\,d},$$

and

$$V_h(0) = \frac{c - \alpha}{1 + c\,\alpha} \quad \text{and} \quad V_h(-\infty) = \frac{c - d}{1 + c\,d}. \tag{5.22}$$

Substituting (5.22) into (5.20) and performing the necessary steps we get

$$\cot\phi = \frac{1+\alpha\, d}{\alpha - d} = \frac{\alpha^2 - \alpha\int_0^\infty \frac{d\tau(t)}{t} + 1}{\int_0^\infty \frac{d\tau(t)}{t}}. \tag{5.23}$$

Taking into account that $\alpha - d > 0$ we combine (5.19), (5.20) with (5.23) and this completes the proof of the theorem. □

Below we will derive the formulas for calculation of the boundary parameter h in the restored Schrödinger operator T_h of the form (5.1). We consider two major cases.

Case 1. In the first case we assume that $\int_0^\infty \frac{d\tau(t)}{t} < \infty$. This means that our function $V(z)$ belongs to the class $SL_{01}^{-1}(R,K)$. In what follows we denote

$$b = \int_0^\infty \frac{d\tau(t)}{t} \quad \text{and} \quad m = m_\infty(-0).$$

Suppose that $b \geq 2$. Then the quadratic inequality (5.16) implies that for all α such that

$$\alpha \in \left(-\infty, \frac{b-\sqrt{b^2-4}}{2}\right] \cup \left[\frac{b+\sqrt{b^2-4}}{2}, +\infty\right) \tag{5.24}$$

the restored operator T_h is accretive. Clearly, this operator is extremal accretive if

$$\alpha = \frac{b \pm \sqrt{b^2-4}}{2}.$$

In particular if $b = 2$ then $\alpha = 1$ and the function

$$V(z) = 1 + \int_0^\infty \left(\frac{1}{t-z} - \frac{1}{t}\right) d\tau(t)$$

is realized using an extremal accretive T_h.

Now suppose that $0 < b < 2$. Then for every $\alpha \in (-\infty, +\infty)$ the restored operator T_h will be accretive and ϕ-sectorial for some $\phi \in (0, \pi/2)$. Consider a function $V(z)$ defined by (5.8). Conducting realizations of $V(z)$ by operators T_h for different values of $\alpha \in (-\infty, +\infty)$ we notice that the operator T_h with the largest angle of sectoriality occurs when

$$\alpha = \frac{b}{2}, \tag{5.25}$$

and is found according to the formula

$$\phi = \arctan\frac{b}{1-b^2/4}. \tag{5.26}$$

This follows from the formula (5.17), the fact that $\alpha^2 - \alpha\, b + 1 > 0$ for all α, and the formula

$$\alpha^2 - \alpha\, b + 1 = \left(\alpha - \frac{b}{2}\right)^2 + \left(1 - \frac{b^2}{4}\right).$$

Now we will focus on the description of the parameter h in the restored operator T_h. It was shown in [6] that the quasi-kernel $\hat{A}$ of the realizing system Θ from theorem 5.5 takes a form

$$\begin{cases} \widehat{A}y = -y'' + qy \\ y'(a) = \eta y(a) \end{cases}, \ \eta = \frac{\mu \mathrm{Re}\, h - |h|^2}{\mu - \mathrm{Re}\, h} \tag{5.27}$$

On the other hand, since $\sigma(t)$ is also the distribution function of the positive self-adjoint operator, we can conclude that $\hat{A}$ equals to the operator $\tilde{B}_\theta$ of the form (5.3). This connection allows us to obtain

$$\theta = \eta = \frac{\mu \mathrm{Re}\, h - |h|^2}{\mu - \mathrm{Re}\, h}. \tag{5.28}$$

Assuming that

$$h = x + iy$$

we will use (5.28) to derive the formulas for x and y in terms of γ. First, to eliminate parameter μ, we notice that (3.15) and (3.5) imply

$$W_\Theta(\lambda) = \frac{\mu - h}{\mu - \overline{h}} \, \frac{m_\infty(\lambda) + \overline{h}}{m_\infty(\lambda) + h} = \frac{1 - iV(z)}{1 + iV(z)}. \tag{5.29}$$

Passing to the limit in (5.29) when $\lambda \to -\infty$ and taking into account that $V(-\infty) = \alpha - b$ and $m_\infty(-\infty) = \infty$ (see [14]) we obtain

$$\frac{\mu - h}{\mu - \overline{h}} = \frac{1 - i(\alpha - b)}{1 + i(\alpha - b)}.$$

Let us denote

$$a = \frac{1 - i(\alpha - b)}{1 + i(\alpha - b)}. \tag{5.30}$$

Solving (5.30) for μ yields

$$\mu = \frac{h - a\bar{h}}{1 - a}.$$

Substituting this value into (5.28) after simplification produces

$$\frac{x + iy - a(x - iy)x - (x^2 + y^2)(1 - a)}{x + iy - a(x - iy) - x(1 - a)} = \theta.$$

After straightforward calculations targeting to represent numerator and denominator of the last equation in standard form one obtains the following relation

$$x - (\alpha - b)\, y = \theta. \tag{5.31}$$

It was shown in [29] that the ϕ-sectoriality of the operator T_h and (5.20) lead to

$$\tan \phi = \frac{\mathrm{Im}\, h}{\mathrm{Re}\, h + m_\infty(-0)} = \frac{y}{x + m_\infty(-0)}. \tag{5.32}$$

Combining (5.31) and (5.32) one obtains

$$x - (\alpha - b)(x \tan \phi + m_\infty(-0)\, \tan \phi) = \theta,$$

or

$$x = \frac{\theta + (\alpha - b)m_\infty(-0)\,\tan\phi}{1 - (\alpha - b)\,\tan\phi}.$$

But $\tan\phi$ is also determined by (5.17). Direct substitution of

$$\tan\phi = \frac{b}{1 + \alpha(\alpha - b)}$$

into the above equation yields

$$x = \theta + \frac{\big[\theta + m_\infty(-0)\big]b(\alpha - b)}{1 + (\alpha - b)^2}.$$

Using the short notation and finalizing calculations we get

$$h = x + iy, \quad x = \theta + \frac{(\alpha - b)[\theta + m]b}{1 + (\alpha - b)^2}, \quad y = \frac{[\theta + m]b}{1 + (\alpha - b)^2}. \tag{5.33}$$

At this point we can use (5.33) to provide analytical and graphical interpretation of the parameter h in the restored operator T_h. Let

$$c = (\theta + m)b.$$

Again we consider three subcases.

Subcase 1. $b > 2$ Using basic algebra we transform (5.33) into

$$(x - \theta)^2 + \left(y - \frac{c}{2}\right)^2 = \frac{c^2}{4}. \tag{5.34}$$

Since in this case the parameter α belongs to the interval in (5.24), we can see that h traces the highlighted part of the circle on Figure 1 as α moves from $-\infty$ towards $+\infty$. We also notice that the removed point $(\theta, 0)$ corresponds to the value of $\alpha = \pm\infty$ while the points h_1 and h_2 correspond to the values $\alpha_1 = \frac{b-\sqrt{b^2-4}}{2}$ and $\alpha_2 = \frac{b+\sqrt{b^2-4}}{2}$, respectively (see Figure 1).

Subcase 2. $b < 2$ For every $\alpha \in (-\infty, +\infty)$ the restored operator T_h will be accretive and ϕ-sectorial for some $\phi \in (0, \pi/2)$. As we have mentioned above, the operator T_h achieves the largest angle of sectoriality when $\alpha = \frac{b}{2}$. In this particular case (5.33) becomes

$$h = x + iy, \quad x = \theta - \frac{2(\theta + m)b^2}{4 + b^2}, \quad y = \frac{4(\theta + m)b}{4 + b^2}. \tag{5.35}$$

The value of h from (5.35) is marked on Figure 2.

Subcase 3. $b = 2$ The behavior of parameter h in this case is depicted on Figure 3. It shows that in this case the function $V(z)$ can be realized using an extremal accretive T_h when $\alpha = 1$. The value of the parameter h according to (5.33) then becomes

$$h = x + iy, \quad x = -m, \quad y = \theta + m. \tag{5.36}$$

Clockwise direction of the circle again corresponds to the change of α from $-\infty$ to $+\infty$ and the marked value of h occurs when $\alpha = 1$.

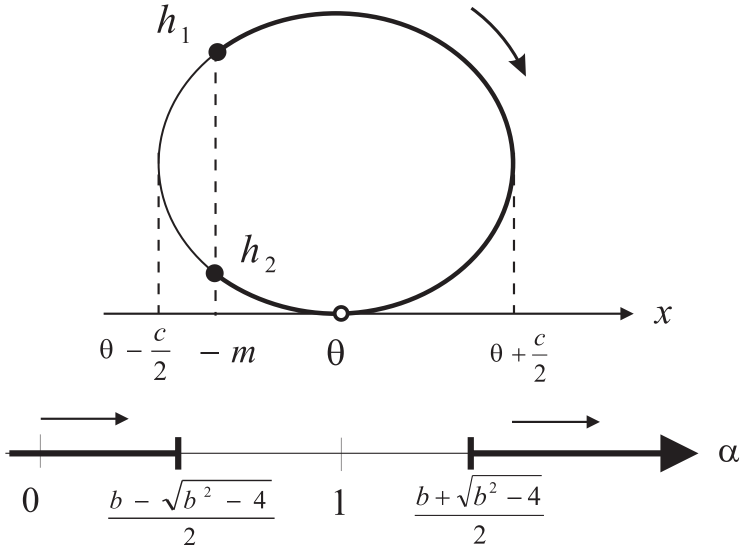

FIGURE 1. $b > 2$

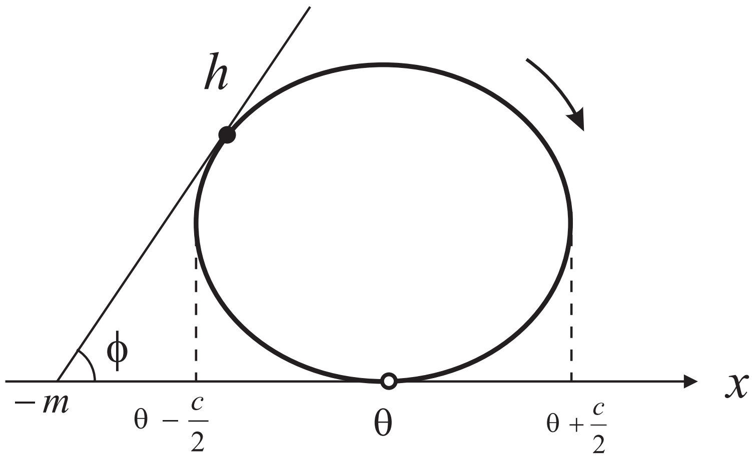

FIGURE 2. $b < 2$

Now we consider the second case.

Case 2. Here we assume that $\int_0^\infty \frac{d\tau(t)}{t} = \infty$. This means that our function $V(z)$ belongs to the class $SL_0^{-1}(R, K)$ and $b = \infty$. According to Theorem 5.7 and formulas (5.16) and (5.17), the restored operator T_h is accretive if and only if

$$\alpha \leq 0,$$

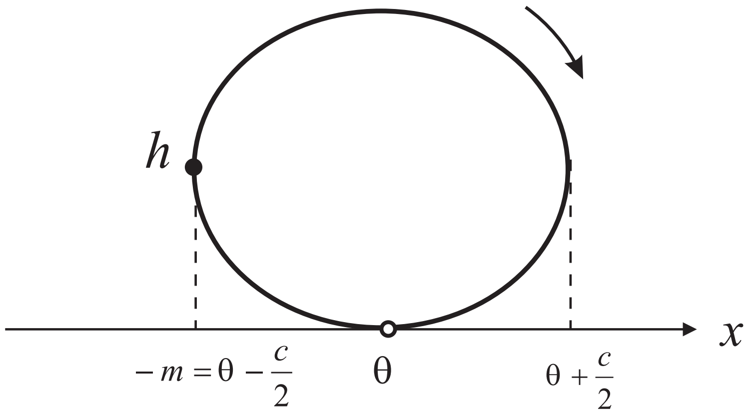

FIGURE 3. $b = 2$

and ϕ-sectorial if and only if $\alpha < 0$. It directly follows from (5.17) that the exact value of the angle ϕ is then found from

$$\tan\phi = -\frac{1}{\alpha}. \tag{5.37}$$

The latter implies that the restored operator T_h is extremal if $\alpha = 0$. This means that a function $V(z) \in SL_0^{-1}(R,K)$ is realized by a system with an extremal operator T_h if and only if

$$V(z) = \int_0^\infty \left(\frac{1}{t-z} - \frac{1}{t}\right) d\tau(t). \tag{5.38}$$

On the other hand since $\alpha \le 0$ the function $V(z)$ is an inverse Stieltjes function of the class $S_0^{-1}(R)$. Applying realization theorems from [13] we conclude that $V(z)$ admits realization by an accumulative system Θ of the form (3.1) with $\mathbb{A}_R$ containing the Friedrichs extension A_F as a quasi-kernel. Here A_F is defined by (2.10). This yields

$$\theta = \frac{\mu x - (x^2 + y^2)}{\mu - x} = \infty, \tag{5.39}$$

and hence $\mu = x$. As in the beginning of the previous case we derive the formulas for x and y, where $h = x + iy$. Assuming that $\alpha \neq 0$ and using (5.32) and (5.37) leads to

$$x = \mu, \quad y = -\frac{x+m}{\alpha}. \tag{5.40}$$

To proceed, we first notice that our function $V(z)$ satisfies the conditions of Theorem 4.9 of [6]. Indeed, the inequality

$$\mu \ge \frac{(\mathrm{Im}\, h)^2}{m_\infty(-0) + \mathrm{Re}\, h} + \mathrm{Re}\, h,$$

that is required to apply this theorem, in our case turns into

$$\mu \geq -\frac{1}{\alpha} + \mu,$$

that is obvious if $\alpha < 0$. Applying Theorem 4.9 of [6] yields

$$\int\limits_0^\infty \frac{d\tau(t)}{1+t^2} = \frac{\operatorname{Im} h}{|\mu - h|^2} \left(\sup_{y \in D(A_F)} \frac{|\mu y(a) - y'(a)|}{\left(\int\limits_a^\infty \left(|y(x)|^2 + |l(y)|^2\right) dx \right)^{\frac{1}{2}}} \right)^2. \tag{5.41}$$

Taking into account that for the case of A_F

$$|\mu y(a) - y'(a)| = |y'(a)|$$

and setting

$$d^{1/2} = \sup_{y \in D(A_F)} \frac{|y'(a)|}{\left(\int\limits_a^\infty \left(|y(x)|^2 + |l(y)|^2\right) dx \right)^{\frac{1}{2}}}, \tag{5.42}$$

we obtain

$$\frac{\operatorname{Im} h}{|\mu - h|^2}\, d = \int\limits_0^\infty \frac{d\tau(t)}{1+t^2}. \tag{5.43}$$

Considering that $\operatorname{Im} h = y$ and $\mu = x$, solving (5.43) for y yields

$$y = \frac{d}{\int_0^\infty \frac{d\tau(t)}{1+t^2}}. \tag{5.44}$$

Consequently, equations (5.40) describing $h = x + iy$ take form

$$x = -m + \frac{\alpha\, d}{\int_0^\infty \frac{d\tau(t)}{1+t^2}}, \quad y = \frac{d}{\int_0^\infty \frac{d\tau(t)}{1+t^2}}. \tag{5.45}$$

The equations (5.45) above provide parametrical equations of the straight horizontal line shown on Figure 4. The connection between the parameters α and h in the accretive restored operator T_h is depicted in bold.

As we mentioned earlier the restored operator T_h is extremal if $\alpha = 0$. In this case formulas (5.45) become

$$x = -m, \quad y = \frac{d}{\int_0^\infty \frac{d\tau(t)}{1+t^2}}. \tag{5.46}$$

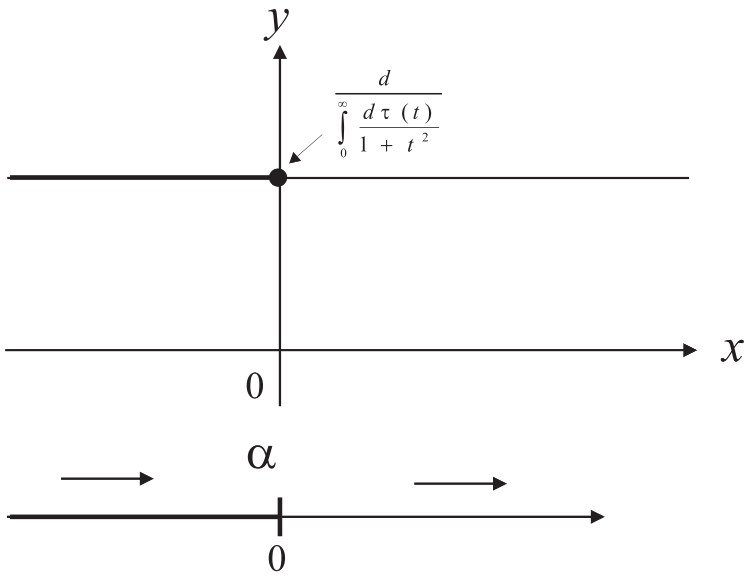

Figure 4. $b = \infty$

6. Realizing systems with Schrödinger operator

Now once we described all the possible outcomes for the restored accretive operator T_h, we can concentrate on the main operator $\mathbb{A}$ of the system (5.15). We recall that $\mathbb{A}$ is defined by formulas (2.6) and beside the parameter h above contains also parameter μ. We will obtain the behavior of μ in terms of the components of our function $V(z)$ the same way we treated the parameter h. As before we consider two major cases dividing them into subcases when necessary.

Case 1. Assume that $b = \int_0^\infty \frac{d\tau(t)}{t} < \infty$. In this case our function $V(z)$ belongs to the class $SL_{01}^{-1}(R, K)$. First we will obtain the representation of μ in terms of x and y, where $h = x + iy$. We recall that

$$\mu = \frac{h - a\bar{h}}{1 - a},$$

where a is defined by (5.30). By direct computations we derive that

$$a = \frac{1 - (\alpha - b)^2}{1 + (\alpha - b)^2} - \frac{2(\alpha - b)}{1 + (\alpha - b)^2} i, \quad 1 - a = \frac{2(\alpha - b)^2}{1 + (\alpha - b)^2} + \frac{2(\alpha - b)}{1 + (\alpha - b)^2} i,$$

and

$$h - a\bar{h} = \left(\frac{2(\alpha - b)^2}{1 + (\alpha - b)^2} x + \frac{2(\alpha - b)}{1 + (\alpha - b)^2} y \right) + \left(\frac{2}{1 + (\alpha - b)^2} y + \frac{2(\alpha - b)}{1 + (\alpha - b)^2} x \right) i.$$

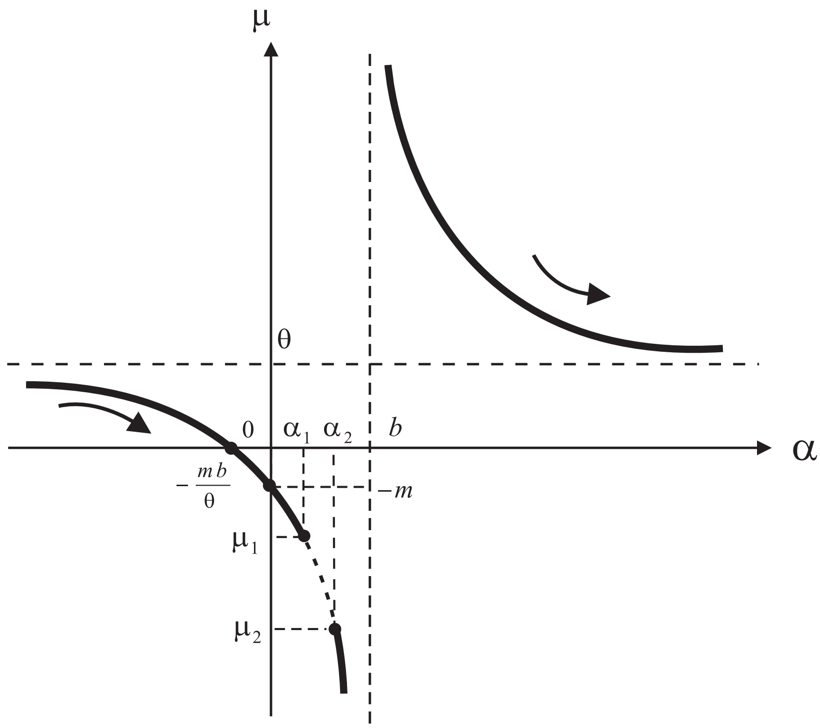

FIGURE 5. $b > 2$

Plugging the last two equations into the formula for μ above and simplifying we obtain

$$\mu = x + \frac{y}{\alpha - b}. \tag{6.1}$$

We recall that during the present case x and y parts of h are described by the formulas (5.33).

Once again we elaborate in three subcases.

Subcase 1. $b > 2$ As we have shown this above, the formulas (5.33) can be transformed into equation of the circle (5.34). In this case the parameter α belongs to the interval in (5.24), the accretive operator T_h corresponds to the values of h shown in the bold part of the circle on Figure 1 as α moves from $-\infty$ towards $+\infty$.

Substituting the expressions for x and y from (5.33) into (6.1) and simplifying we get

$$\mu = \theta + \frac{(\theta + m)b}{\alpha - b}. \tag{6.2}$$

The connection between values of α and μ is depicted on Figure 5.

We note that $\mu = 0$ when $\alpha = -\frac{mb}{\theta}$. Also, the endpoints

$$\alpha_1 = \frac{b - \sqrt{b^2 - 4}}{2} \quad \text{and} \quad \alpha_2 = \frac{b + \sqrt{b^2 - 4}}{2}$$

of α-interval (5.24) are responsible for the μ-values

$$\mu_1 = \theta + \frac{(\theta + m)b}{\alpha_1} \quad \text{and} \quad \mu_2 = \theta + \frac{(\theta + m)b}{\alpha_2}.$$

The values of μ that are acceptable parameters of operator $\mathbb{A}$ of the restored system with an accretive operator T_h make the bold part of the hyperbola on Figure 5. It follows from Theorems 4.4 and 4.4 that the operator $\mathbb{A}$ of the form (2.6) is accumulative if and only if $\alpha \leq 0$ and thus μ belongs to the part of the left branch on the hyperbola where $\alpha \in (-\infty, 0]$. We note that Figure 5 shows the case when $-m < 0$, $\theta > 0$, and $\theta > -m$. Other possible cases, such as $(-m < 0,\ \theta < 0,\ \theta > -m)$, $(-m < 0,\ \theta = 0)$, and $(m = 0,\ \theta > 0)$ require corresponding adjustments to the graph shown in the picture 5.

Subcase 2. $b < 2$ For every $\alpha \in (-\infty, +\infty)$ the restored operator T_h will be accretive and ϕ-sectorial for some $\phi \in (0, \pi/2)$. As we have mentioned above, the operator T_h achieves the largest angle of sectoriality when $\alpha = \frac{b}{2}$. In this particular case (5.33) becomes (5.35). Substituting $\alpha = b/2$ and (5.35) into (6.1) we obtain

$$\mu = -(\theta + 2m). \tag{6.3}$$

This value of μ from (6.3) is marked on Figure 6. The corresponding operator $\mathbb{A}$ of the realizing system is based on these values of parameters h and μ.

Subcase 3. $b = 2$ The behavior of parameter μ in this case is also shown on Figure 6. It was shown above that in this case the function $V(z)$ can be realized using an extremal accretive T_h when $\alpha = 1$. The values of the parameters h and μ then become

$$h = x + iy, \quad x = -m, \quad y = \theta + m, \quad \mu = -(\theta + 2m).$$

The value of μ above is marked on the left branch of the hyperbola and occurs when $\alpha = 1 = b/2$.

Case 2. Again we assume that $\int_0^\infty \frac{d\tau(t)}{t} = \infty$. Hence $V(z) \in SL_0^{-1}(R, K)$ and $b = \infty$. As we mentioned above the restored operator T_h is accretive if and only if $\alpha \leq 0$ and ϕ-sectorial if and only if $\alpha < 0$. It is extremal if $\alpha = 0$. The values of x and y, were already calculated and are given in (5.45). In particular, the value for μ is given by

$$\mu = x = -m + \frac{\alpha\, d}{\int_0^\infty \frac{d\tau(t)}{1+t^2}}. \tag{6.4}$$

where d is defined in (5.42). Figure 7 gives graphical representation of this case. The left bold part of the line corresponds to the values of μ that yield an accumulative realizing system. If $m = 0$ then the line passes through the origin and the graph

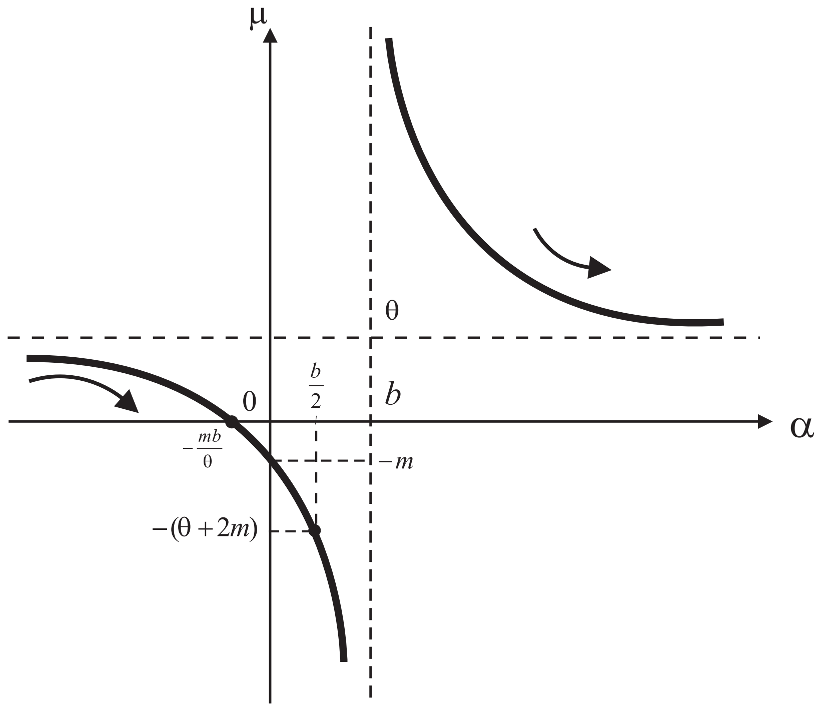

FIGURE 6. $b < 2$ and $b = 2$

should be adjusted accordingly. In the case when $\alpha = 0$ and T_h is extremal we have $\mu = m$.

Example

We conclude this paper with simple illustration. Consider a function

$$V(z) = i\sqrt{z}. \tag{6.5}$$

A direct check confirms that $V(z)$ is an inverse Stieltjes function. It can be shown (see [25] pp. 140–142) that the inversion formula

$$\tau(\lambda) = C + \lim_{y \to 0} \frac{1}{\pi} \int_0^\lambda \operatorname{Im}\left(i\sqrt{x+iy}\right)\, dx \tag{6.6}$$

describes the distribution function for a self-adjoint operator

$$\begin{cases} \tilde{B}_\infty y = -y'' \\ y(0) = 0. \end{cases}$$

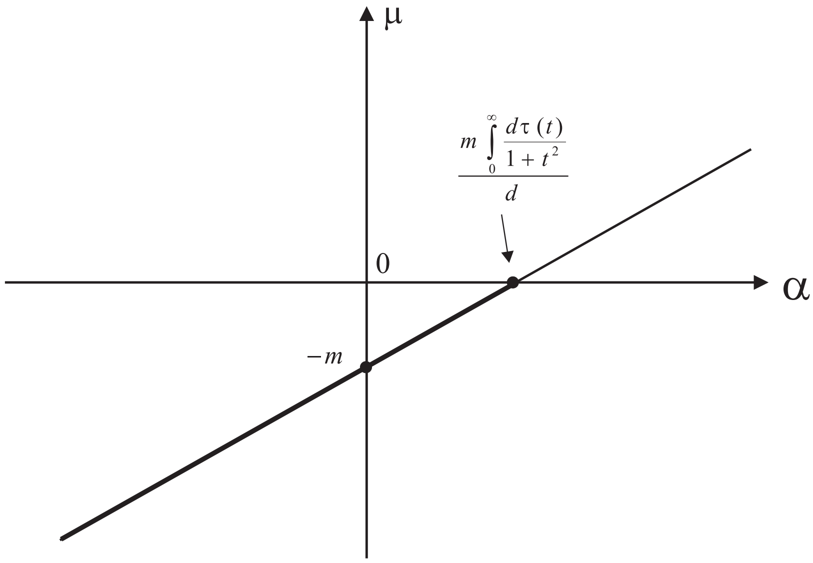

FIGURE 7. $b = \infty$

The corresponding to $\tilde{B}_\infty$ symmetric operator is

$$\begin{cases} B_\infty y = -y'' \\ y(0) = y'(0) = 0. \end{cases} \tag{6.7}$$

It was also shown in [25] that $\tau(\lambda) = 0$ for $\lambda \le 0$ and

$$\tau'(\lambda) = \frac{1}{\pi}\sqrt{\lambda} \quad \text{for} \quad \lambda > 0. \tag{6.8}$$

By direct calculations one can confirm that

$$V(z) = \int_0^\infty \left(\frac{1}{t-z} - \frac{1}{t}\right) d\tau(t) = i\sqrt{z},$$

and that

$$\int_0^\infty \frac{d\tau(t)}{t} = \int_0^\infty \frac{dt}{\pi\sqrt{t}} = \infty.$$

It is also clear that the constant term in the integral representation (5.7) is zero, i.e., $\alpha = 0$.

Let us assume that $\tau(t)$ satisfies our definition of spectral distribution function of the pair B_∞, $\tilde{B}_\infty$ given in Section 5. Operating under this assumption, we proceed to restore parameters h and μ and apply formulas (5.45) for the values $\alpha = 0$ and $m = m_\infty(-0) = 0$ (see [6]). This yields $x = 0$. To obtain y we first find the value of

$$\int_0^\infty \frac{d\tau(t)}{1+t^2} = \frac{1}{\sqrt{2}},$$

and then use formula (5.42) to get the value of d. This yields $d = 1/\sqrt{2}$. Consequently,

$$y = \frac{d}{\int_0^\infty \frac{d\tau(t)}{1+t^2}} = 1,$$

and hence $h = yi = i$. From (6.4) we have that $\mu = 0$ and (2.6) becomes

$$\mathbb{A}\, y = -y'' - [iy'(0) + y'(0)]\delta'(x). \tag{6.9}$$

The operator T_h in this case is

$$\begin{cases} T_h y = -y'' \\ y'(0) = iy(0). \end{cases}$$

The channel vector g of the form (3.11) then equals

$$g = \delta'(x), \tag{6.10}$$

satisfying

$$\operatorname{Im}\mathbb{A} = \frac{\mathbb{A} - \mathbb{A}^*}{2i} = KK^* = (.,g)g,$$

and channel operator $Kc = cg$, $(c \in \mathbb{C})$ with

$$K^* y = (y, g) = y'(0). \tag{6.11}$$

The real part of $\mathbb{A}$

$$\operatorname{Re}\mathbb{A}\, y = -y'' - y(0)\delta'(x)$$

contains the self-adjoint quasi-kernel

$$\begin{cases} \widehat{A} y = -y'' \\ y(0) = 0. \end{cases}$$

A system of the Livšic type with Schrödinger operator of the form (5.15) that realizes $V(z)$ can now be written as

$$\Theta = \begin{pmatrix} \mathbb{A} & K & 1 \\ \mathcal{H}_+ \subset L_2[a, +\infty) \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix}.$$

where $\mathbb{A}$ and K are defined above. Now we can back up our assumption on $\tau(t)$ to be the spectral distribution function of the pair B_∞, $\tilde{B}_\infty$. Indeed, calculating the function $V_\Theta(z)$ for the system Θ above directly via formula (3.16) with $\mu = 0$ and comparing the result to $V(z)$ gives the exact value of $h = i$. Using the uniqueness of the unitary mapping U in the definition of spectral distribution function (see Remark 5.6 of [14]) we confirm that $\tau(t)$ is the spectral distribution function of the pair B_∞, $\tilde{B}_\infty$.

Remark 6.1. All the derivations above can be repeated for an inverse Stieltjes-like function

$$V(z) = \alpha + i\sqrt{z}, \qquad -\infty < \alpha < +\infty,$$

with very minor changes. In this case the restored values for h and μ are described as follows:

$$h = x + iy, \quad x = \alpha, \quad y = 1, \quad \mu = \alpha.$$

The dynamics of changing h according to changing α is depicted on Figure 4 where the horizontal line has a y-intercept of 1. The behavior of μ is described by a sloped line $\mu = \alpha$ (see Figure 7 with $m = 0$). In the case when $\alpha \leq 0$ our function becomes inverse Stieltjes and the restored system Θ is accretive. The operators $\mathbb{A}$ and K of the restored system are given according to the formulas (2.6) and (3.13), respectively.

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