REALIZATION AND FACTORIZATION PROBLEMS FOR J-CONTRACTIVE OPERATOR-VALUED FUNCTIONS IN HALF-PLANE AND SYSTEMS WITH UNBOUNDED OPERATORS

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In this paper realization problems for operator-valued R-functions acting in finite-dimentional Hilbert space E as linear-fractional transformations of the transfer operator-functions of linear stationary conservative dynamic systems (l.s.c.d.s.) θ of the form

$$(0)\qquad \begin{cases} (\mathbb{A}-zI)=KJ\phi_- \\ \phi_+=\phi_- -2iK^*x \end{cases} \qquad (\mathrm{Im}\ \mathbb{A}=KJK^*)$$

are investigated. In a system θ of the form (0) an operator $\mathbb{A}$ is a bounded linear operator, acting from $\mathfrak{H}_+$ into $\mathfrak{H}_-$,$\mathfrak{H}_+\subset\mathfrak{H}\subset\mathfrak{H}_-$ is rigged Hilbert space,

$$\mathbb{A}\supset T\supset A,\ \mathbb{A}^*\supset T^*\supset A,$$

where A is Hermitian operator in $\mathfrak{H}, T$ is nonhermitian operator in $\mathfrak{H}$, K is a linear bounded operator from E into $\mathfrak{H}, J=J^*=J^{-1}$ and this operator is acting in $E, \phi_\pm\in E$, ϕ_- is an input vector, ϕ_+ is an output vector, $x\in\mathfrak{H}_+$ is a vector of an inner state of the system θ, an operator-valued function

$$W_\theta(z)=I-2iK^*(\mathbb{A}-zI)^{-1}KJ \qquad (\phi_+=W_\theta(z)\phi_-)$$

is a transfer operator-function of the system θ.

It turns out, that not all operator-valued R-functions can be realized in the above mentioned sense and we give a criteria of such a realizability in this paper.In terms of realizable operator-valued R-functions we specialize in subclasses of the following types:

(1) a subclass for which $\overline{\mathfrak{D}(A)}=\mathfrak{H},\ \mathfrak{D}(T)\neq\mathfrak{D}(T^*)$
(2) a subclass for which $\overline{\mathfrak{D}(A)}\neq\mathfrak{H},\ \mathfrak{D}(T)=\mathfrak{D}(T^*)$
(3) a subclass for which $\overline{\mathfrak{D}(A)}\neq\mathfrak{H},\ \mathfrak{D}(T)\neq\mathfrak{D}(T^*)$

Given classes of operator-valued R-functions allow us to define classes of J-contractive operator-valued functions in half-plane, which can be realized as a transfer mapping of the system θ with, generally speaking, unbounded main operator.A problem when the product of J-contractive operator-valued functions from defined classes belongs to the same classes , is investigated.

We consider also a problem of factorization of a realizable J-contractive operator-function in half-plane which is connected with invariant subspaces of the main operator T of a system θ.

A class of a realizable J-contractive operator-valued functions in half-plane for which main operators T of systems θ have a property 1) turned out to be very interesting.The theorem on constant J-unitary factor in which we show that product (in any order) of an arbitrary constant J-unitary operator and J-contractive operator-valued function in a half-plane from the mentioned above class belongs to this class, takes place.

We investigate also a problem of connection between realizations of two transfer mappings differing in the constant J-unitary factor.

Note that for the first time the problem of studying oscillations in lengthy line with the aid of system theory with unbounded operators had been formulated by M.S.Livšic [11] and later but independently - by J.W.Helton [7].

2.

Let A be a linear closed Hermitian operator, acting in Hilbert space $\mathfrak{H}$ with, generally speaking, non-dense domain $\mathfrak{D}(A)$. Let $\mathfrak{H}_0 = \overline{\mathfrak{D}(A)}$, A^*-conjugate to A (we consider operator A as acting from $\mathfrak{H}_0$ into $\mathfrak{H}$). Let us denote $\mathfrak{H}_+ = \mathfrak{D}(A^*)$ $\quad (\overline{(\mathfrak{D}(A^*)} = \mathfrak{H})$ and define in $\mathfrak{H}_+$ scalar product

$$(f,g)_+ = (f,g) + (A^*f, A^*g) \qquad (f,g \in \mathfrak{H}_+)$$

and then build the rigged Hilbert space $\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_-$. See [1,2]. We call an operator A regular, if PA is a closed operator in $\mathfrak{H}_0$ (P is an orthoprojector $\mathfrak{H}$ onto $\mathfrak{H}_0$). A regular operator A is called O-operator if its semidefect numbers (defect numbers of an operator PA) are equal to zero.

An operator $\mathbb{A} \in [\mathfrak{H}_+, \mathfrak{H}_-]$ $\quad ([\mathfrak{H}_+, \mathfrak{H}_-]$ - the set of all linear bounded operators acting from $\mathfrak{H}_+$ into $\mathfrak{H}_-$) is called biextension of a regular Hermitian operator A, if

$$\mathbb{A} \supset A, \quad \mathbb{A}^* \supset A$$

If $\mathbb{A} = \mathbb{A}^*$, then $\mathbb{A}$ is called a selfadjoint biextension of an operator A. Note, that identifying the space conjugate to $\mathfrak{H}_\pm$ with $\mathfrak{H}_\pm$. We have that $\mathbb{A}^* \in [\mathfrak{H}_+, \mathfrak{H}_-]$.

We say that the closed linear operator T with dense domain in $\mathfrak{H}$ belongs to the class Λ_A if:

(1) $T \supset A, \quad T^* \supset A$ where A is a maximal common Hermitian part of T and T^* and operator A is regular.

(2) $(-i)$ is a regular point of T.[1]

[1]The condition, that $(-i)$ is a regular point in the definition of the class Λ_A is non-essential. It is sufficient to require the existence of some regular point for T.

An operator $\mathbb{A} \in [\mathfrak{H}_+, \mathfrak{H}_-]$ is called a $(*)$-extension of an operator T of the class Λ_A if

$$\mathbb{A} \supset T \supset A$$

$$\mathbb{A}^* \supset T^* \supset A$$

This $(*)$-extension $\mathbb{A}$ of an operator T is called correct, if an operator

$$\mathbb{A}_R = \frac{1}{2}(\mathbb{A} + \mathbb{A}^*)$$

has the following property: An operator

$$\hat{A}f = \mathbb{A}_R f$$

$$\mathfrak{D}(\hat{A}) = \{f \in \mathfrak{H}_+ : \mathbb{A}_R f \in \mathfrak{H}\}$$

is a selfadjoint operator in Hilbert space $\mathfrak{H}$.

Definition 1.. *The aggregate*

$$\theta = \begin{pmatrix} \mathbb{A} & K & J \\ \mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_- & & E \end{pmatrix} \tag{1}$$

is called a linear stationary conservative dynamic system if

(1) $\mathbb{A}$ *is a correct $(*)$-extension of an operator T of the class Λ_A.*
(2) $J = J^* = J^{-1} \in [E, E]$, $\quad \dim E < \infty$
(3) $\mathbb{A} - \mathbb{A}^* = 2iKJK^*$, *where* $K \in [E, \mathfrak{H}_-] \qquad (K^* \in [\mathfrak{H}_+, E])$

In addition an operator K is called a channel operator and J is called a direction operator. System θ of the form (1) will be called a passage system if $J \neq I$ and a scattering system if $J = I$.

3.

As it is known [8] an operator-function $V(z) \in [E, E]$ is called an operator-valued R-function if it is holomorphic in the upper half-plane and $\operatorname{Im} V(z) \geq 0$ when $\operatorname{Im} z > 0$.

An operator-valued R-function, acting in Hilbert space $E(\dim E < \infty)$ has, as it is known [8], integral representation

$$V(z) = Q + F \cdot z + \int_{-\infty}^{+\infty} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) dG(t) \tag{2}$$

where $Q = Q^*$, $F \geq 0$ in the Hilbert space E, $G(t)$ is non-decreasing operator-function on $(-\infty, +\infty)$ for which

$$\int_{-\infty}^{+\infty} \frac{dG(t)}{1 + t^2} < \infty.$$

Definition 2. *We call an operator-valued R-function acting in Hilbert space* E *(*$\dim E < \infty$*) realizable if in some neighbourhood of point* $(-i)$ $V(z)$ *can be represented in the form*

$$V(z) = i[W_\theta(z) + I]^{-1}[W_\theta(z) - I]J \tag{3}$$

where $W_\theta(z)$ *is a transfer operator-function of some l.s.c.d.* θ *with the direction operator* J $(J = J^* = J^{-1} \in [E, E])$.

It may be shown, that the transfer operator-function of the system θ of the form (1) has the following properties:

$$\begin{aligned} &W_\theta^*(z)JW_\theta(z) - J \geq 0 \quad (\operatorname{Im} z > 0, z \in \rho(T)) \\ &W_\theta^*(z)JW_\theta(z) - J = 0 \quad (\operatorname{Im} z = 0, z \in \rho(T)) \\ &W_\theta^*(z)JW_\theta(z) - J \leq 0 \quad (\operatorname{Im} z < 0, z \in \rho(T)) \end{aligned} \tag{4}$$

where $\rho(T)$ is the set of regular points of an operator T.

Similar relations take place if we change $W_\theta(z)$ on to $W_\theta^*(z)$ in (4). Thus, a transfer operator- function of the system θ of the form (1) is J-contractive in the lower half-plane on the set of regular points of an operator T and J-unitary on real regular points of an operator T. Let θ be a l.s.c.d.s of the form (1). We consider an operator-function

$$V_\theta(z) = K^*(\mathbb{A}_R - zI)^{-1}K \tag{5}$$

The transfer operator-function $W_\theta(z)$ of the system θ and an operator-function $V_\theta(z)$ of the form (5) are connected with relation

$$V_\theta(z) = i[W_\theta(z) + I]^{-1}[W_\theta(z) - I]J \tag{6}$$

Operator T of the class Λ_A is called completely nonselfadjoint [4], [10] if there exists no reducing invariant subspace on which one induces a self-adjoint operator.

Realization of an operator-valued R-function $V(z) \in [E, E]$ by the system θ of the form (1) is called minimal if an operator T is completely nonselfadjoint.

Definition 3. *An operator-valued R-function* $V(z) \in [E, E]$ $(\dim E < \infty)$ *will be said to be a member of the class* $N(R)$ *if in the representation* (2)

$$\begin{aligned} &i) \qquad F = 0, \\ &ii) \qquad Qe = \int_{-\infty}^{+\infty} \frac{dG(t)}{1+t^2} e \end{aligned}$$

for all $e \in E$*, when*

$$\int_{-\infty}^{+\infty} (dG(t)e, e)_E < \infty$$

Theorem 1. . *Let* θ *be a l.s.c.d.s. of the form (1),* $\dim E < \infty$*. Then operator-function* $V_\theta(z)$ *of the form (5), (6) belongs to the class* $N(R)$*. Conversely, let operator-function* $V(z)$ *act in a finitedimensional Hilbert space* E *and belong to the class* $N(R)$*. Then it admits minimal realization by the system* θ *of the form (1) with a preassigned direction operator* $J(J = J^* = J^{-1} \in [E, E])$.

From here, in particular, it follows, that every operator-function of the class $N(R)$ acting in finitedimensional Hilbert space admits minimal realization with the help of some scattering system θ of the form (1).

Definition 4. . *An operator-valued R-function $V(z) \in [E,E] \quad (\dim E < \infty)$ of the class $N(R)$ is said to be a member of the subclass $N_0(R)$ if in the representation (2)*

$$\int_{-\infty}^{+\infty} (dG(t)e,e)_E = \infty \qquad (e \in E, e \neq 0)$$

From here it follows, that operator-function $V(z)$ of the class $N_0(R)$ has a representation

$$V(z) = Q + \int_{-\infty}^{+\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) dG(t) \qquad (Q = Q^*)$$

Note, that an operator Q can be an arbitrary self-adjoint operator in Hilbert space E.

Definition 5. . *An operator-valued R-function $V(z) \in [E,E] \quad (\dim E < \infty)$ of the class $N(R)$ is said to be a member of the subclass $N_1(R)$ if in the representation (2)*

$$\int_{-\infty}^{+\infty} (dG(t)e,e)_E < \infty \qquad (e \in E)$$

Thus, an operator-function $V(z)$ of the class $N_1(R)$ has a representation

$$V(z) = \int_{-\infty}^{+\infty} \frac{1}{t-z}\, dG(t)$$

An operator-valued R-function $V(z) \in [E,E], \qquad (\dim E < \infty)$ of the class $N(R)$ will be said to be a member of the subclass $N_{01}(R)$ if the subspace

$$E_{\infty}^{\perp} = \left\{ e \in E : \int_{-\infty}^{+\infty} (dG(t)e,e)_E < \infty \right\}$$

possess a property: $E_{\infty}^{\perp} \neq \emptyset, \quad E_{\infty}^{\perp} \neq E$.

Theorem 2. *Let θ be a l.s.c.d. of the form (1), $\dim E < \infty$ where A is a linear closed Hermitian operator with a dense domain and $\mathfrak{D}(T) \neq \mathfrak{D}(T^*)$.Then operator-function $V_\theta(z)$ of the form (5),(6) belongs to the class $N_0(R)$.Conversely, let an operator-function $V(z)$ act in a finitedimensional Hilbert space E and belong to the class $N_0(R)$. Then it admits a minimal realization by the system θ of the form (1) with a preassigned direction operator $J(J = J^n = J^{-1} \in [E,E])$ and A is a linear closed Hermitian operator with dense domain, $\mathfrak{D}(T) \neq \mathfrak{D}(T^*)$.*

Theorem 3. *Let θ be a l.s.c.d.s. of the form (1) where $\mathfrak{D}(T) = \mathfrak{D}(T^*)$ and A is a linear closed regular Hermitian O-operator. Then an operator-function $V_\theta(z)$ of the form (5), (6) belongs to the class $N_1(R)$. Conversely, let an operator-function $V(z)$ act in a finitedimensional Hilbert space E and belong to the class $N_1(R)$. Then it admits a minimal realization by the system θ of the form (1) with a preassigned direction operator $J(J = J^* = J^{-1} \in [E,E])$ where A is a linear closed regular Hermitian O-operator with a non-dense domain and $\mathfrak{D}(T) = \mathfrak{D}(T^*)$.*

The theorem, similar to theorems (1)-(3), takes place also for an operator-functions of the class $N_{01}(R)$.

Definition 6. *An operator-function $W(z)$ acting in finitedimensional Hilbert space E is said to be a member of the class $\Omega(R,J)$ $\quad ,(\Omega_1(R,J),\Omega_{01}(R,J))$, where $J = J^* = J^{-1} \in [E,E]$, if it is holomorphic in some neighbourhood of the point $(-i)$ and an operator-function*

$$V(z) = i[W(z)+I]^{-1}[W(z)-I]J$$

belongs to the class $N(R)\,(N_0(R), N_1(R), N_{01}(R))$ respectively.

Thus, classes $\Omega(R,J), \Omega_0(R,J), \Omega_1(R,J), \Omega_{01}(R,J)$ represent the set of J-contructive operator-functions in the lower half-plane with properties (4), which can be realized as transfer operator-functions of the system θ of the form (1) with those or other characteristics of the main operator T.

The theorems (1)-(3) develop and make precise the known results by M.S.Livšic on the theory of inverse problem of the characteristic operator-functions and systems [10]. For the considered class of operators the theorem 3 reinforces one result from [9].

Theorem 4. *Let operator-functions $W_1(z)$ and $W_2(z)$ acting in finitedimensional Hil-*
bert space E belong to classes $\Omega(R,J)\,(\Omega_0(R,J),\Omega_1(R,J))$. Then their product (in any order) also belongs to classes $\Omega(R,J)\,(\Omega_0(R,J),\Omega_1(R,J))$ respectively.

Theorem 5. *Let*

$$(7)\quad \theta_1 = \begin{pmatrix} \mathbb{A}_1 & K_1 & J \\ \mathfrak{H}_{+1} \subset \mathfrak{H}_1 \subset \mathfrak{H}_{-1} & & E \end{pmatrix} \text{ and } \theta_2 = \begin{pmatrix} \mathbb{A}_2 & K_2 & J \\ \mathfrak{H}_{+2} \subset \mathfrak{H}_2 \subset \mathfrak{H}_{-2} & & E \end{pmatrix}$$

be linear stationary conservative dynamic systems of the form (1) so that their transfer operator-functions $W_{\theta_1}(z)$ and $W_{\theta_2}(z)$ belong to the class $\Omega_{01}(R,J)$. The product $W_{\theta_1}(z)\cdot W_{\theta_2}(z)$ $\quad (W_{\theta_2}(z)\cdot W_{\theta_1}(z))$ also belongs to the class $\Omega_{01}(R,J)$ if and only if the set

$$\mathfrak{D}_{12} = \{x = x_1 + x_2 : x_1 \in \mathfrak{D}(T_1^*),\, x_2 \in \mathfrak{D}(T_2), K_1^* x_1 + K_2^* x_2 = 0\}$$

and respectively

$$\mathfrak{D}_{21} = \{x = x_2 + x_1 : x_2 \in \mathfrak{D}(T_2^*),\, x_1 \in \mathfrak{D}(T_1,\, K_2^* x_2 + K_1^* x_1 = 0\}$$

is non-dense in Hilbert space $\mathfrak{H}_1 \otimes \mathfrak{H}_2$.

There exists an example of the systems θ_1 and θ_2 transfer mappings of which $W_{\theta_1}(z)$ and $W_{\theta_2}(z)$ belong to the class $\Omega_{01}(R,J)$, but their product $W_{\theta_1}(z)\cdot W_{\theta_2}(z)$ belongs to the class $\Omega_0(R,J)$.The criterion when $W_{\theta_1}(z)\cdot W_{\theta_2}(z)$ $\quad (W_{\theta_2}(z)\cdot W_{\theta_1}(z))$ belongs to the class $\Omega_{01}(R,J)$ if $W_{\theta_1}(z)$ belongs to the $\Omega_0(R,J)$ and $W_{\theta_2}(z)$ belongs to the $\Omega_1(R,J)$ is found. It may be shown also, that if $W_1(z) \in \Omega_0(R,J), W_2(z) \in \Omega_1(R,J)$ and these operator-functions, acting in finitedimensional Hilbert space E, are commuting, then $W_1(z)W_2(z) \in \Omega_{01}(R,J)$.

Note, that theorem 4 permits system θ, the transfer mapping of which $W_\theta(z) = W_{\theta_1}(z)\cdot W_{\theta_2}(z)$, to be built constructively, when transfer mappings $W_{\theta_1}(z)$ and $W_{\theta_2}(z)$ of systems θ_1 and θ_2 of the form (1) are known.

There is a procedure of "projection" of the system θ of the form (1) onto an arbitrary invariant subspace of the operator T and its orthogonal complement.In addition, for systems θ_1 and θ_2, which are "projections" of the system θ a factorization formula

$$W_\theta(z) = W_{\theta_1}(z) \cdot W_{\theta_2}(z) \tag{8}$$

is valid.Besides, if a transfer mapping $W_\theta(z)$ belongs to the class $\Omega(R, J)$, then $W_{\theta_1}(z)$ and $W_{\theta_2}(z)$ also belong to the class $\Omega(R, J)$. The analogous property of factorization takes place for transfer mappings $W_\theta(z)$ of the class $\Omega_1(R, J)$. But if $W_\theta(z)$ belongs to the class $\Omega_0(R, J)$ or $\Omega_{01}(R, J)$, then in the factorization formula (8), as it follows from theorem 5 and comments to it, factors $W_{\theta_1}(z)$ and $W_{\theta_2}(z)$ can, generally speaking, be in different classes $(\Omega_0(R, J), \Omega_1(R, J), \Omega_{01}(R, J))$.

Theorem 6. *Let θ be a l.s.c.d.s. of the form (1) with an invertible channel operator K and a direction operator J $\quad (J = J^* = J^{-1} \in [E, E], \dim E < \infty)$, where transfer mapping $W_\theta(z)$ belongs to the class $\Omega_0(R, J)$. Then for arbitrary constant J-unitary operators B and C, acting in Hilbert space E, the product $B \cdot W_\theta(z) \cdot C$, also belongs to the class $\Omega_0(R, J)$.*

Theorem 6 in somewhat other wording was established by Yu.M.Arlinskiĭ and E.R.Tsekanovskiĭ [1] and being published for the first time. Note that theorem 6 fails to be true if $\dim E = \infty$. See [1]. There is a procedure of realization $W(z) \cdot C \quad (B \cdot W(z))$ knowing realization $W(z) \in \Omega_0(R, J)$, where B and C are an arbitrary constant J-unitary operator [1]. Besides, it was established [1] that if $W_{\theta_1}(z)$ and $W_{\theta_2}(z)$ belong to the class $\Omega_0(R, J)$ and operators $\mathbb{A}_1$ and A_2 of systems θ_1 and θ_2 of the form (7) are different correct $(*)$-extensions of the same operator T of the class Λ_A, then under some restrictions of the channel operators K_1 and K_2 of systems θ_1 and θ_2, respectively, transfer mappings W_{θ_1} and W_{θ_2} satisfy the relation

$$W_{\theta_2}(z) = W_{\theta_1}(z) \cdot C$$

where C is some constant J-unitary operator.

Note that theorems 1 - 6 are a further development and complement of the investigations by M.S.Brodskiĭ, M.S.Livšic, V.P.Potapov, A.V.Shtraus, N.Bart, I.Gohberg, M.Kaashoek, A.C.Ran [3] [4] [6] [10] [12] [13,14] (see, also survey [5]). Realization problems for a very general class of transfer functions of systems with, generally speaking, unbounded operators have recently been investigated by G.Weiss [15] .

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