On Krein's Formula in the Case of Non-densely Defined Symmetric Operators

Sergey Belyi

Department of Mathematics
Troy State University
Troy, AL 36082, USA
E-mail: sbelyi@trojan.troyst.edu

Govind Menon

Department of Mathematics
Troy State University
Troy, AL 36082, USA
E-mail: gmenon@trojan.troyst.edu

Eduard Tsekanovskii

Department of Mathematics
Niagara University, NY 14109, USA
E-mail: tsekanov@niagara.edu

In this paper we provide some additional results related to Krein's resolvent formula for a non-densely defined symmetric operator. We show that coefficients in Krein's formula can be expressed in terms of analogues of the classical von Neumann formulas. The relationship between two Weyl-Tichmarsh m-functions corresponding to self-adjoint extensions of a non-densely defined symmetric operator is established.

1. INTRODUCTION

The goal of this paper is to provide additional results in connection with Krein's formula. In the recent paper by F. Gesztesy, K. Makarov, and E. Tsekanovskii [11], the authors have revisited Krein's formula associated with self-adjoint extensions of a densely-defined symmetric operator. They showed that the coefficients in Krein's formula can be expressed in terms of the classical von Neumann parametrization formulas.

In this paper, we extend the above-mentioned results to the case when the original operator is *not* densely defined. The analogues of the von Neumann formulas to the non-dense case, also known as Krasnoselskii's formulas

[13], take a form of indirect decompositions. Besides the Krasnoselskii formulas there is another analogue of the von Neumann formulas obtained by Yu. Arlinskii, Ju. Šmuljan, and E. Tsekanovskii [4], [21], where these decompositions are direct.

In this note, we show that the coefficients of the Krein resolvent formula can be expressed in terms of both indirect and direct analogues of the von Neumann formulas mentioned above. In order to treat the direct decomposition case, we introduce triplets of Hilbert spaces and associated projection operators. The concept of the operator-valued Weyl-Tichmarsh m-function is extended to the case of the non-densely defined operator and its self-adjoint extensions. Here we show that, in the case of indirect decomposition, this result resembles the case of a symmetric densely-defined operator [11] unlike the direct decomposition case. We should mention that the linear-fractional transformation between two m-functions for an operator with deficiency indices $(1,1)$ was studied in great detail by Aronszajn [5] and Donoghue [9].

We conclude our discussion with two examples. In the first example, the Hilbert space is finite dimensional, and consequently, the semi- deficiency spaces are trivial. In the final example, the deficiency indices are (∞,∞), but the semi-deficiency spaces are one-dimensional. The coefficient function and its connections to the generalizations of the von Neumann formula are explicitly calculated for both examples.

In this paper we follow the notation of [11].

2. PRELIMINARY RESULTS

In this section we recall some basic facts related to the theory of extensions of linear operators with non-dense domain.

Let $\mathcal{H}$ denote a Hilbert space with inner product (x,y) and let $\mathcal{B}(\mathcal{H})$ be the Banach space of bounded linear operators on $\mathcal{H}$. Let $\dot{A}$ be a closed linear symmetric operator ($(\dot{A}x,y)=(x,\dot{A}y)$, $\forall x,y\in\mathfrak{D}(\dot{A})$), acting in the Hilbert space $\mathcal{H}$ with generally speaking a non-dense domain $\mathfrak{D}(\dot{A})$. Let $\mathcal{H}_0=\overline{\mathfrak{D}(\dot{A})}$, and $\dot{A}^*$ be the adjoint to the operator $\dot{A}$ (we consider $\dot{A}$ acting from $\mathcal{H}_0$ into $\mathcal{H}$).

It is easy to see that for the symmetric operator $\dot{A}$, $\mathfrak{D}(\dot{A})\subset\mathfrak{D}(\dot{A}^*)$, and $\dot{A}^*y=P\dot{A}y$ ($\forall y\in\mathfrak{D}(\dot{A})$), where P is an orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_0$. We put

$$\mathfrak{L}:=\mathcal{H}\ominus\mathcal{H}_0\quad \mathfrak{M}_\lambda:=(\dot{A}-\lambda I)\mathfrak{D}(\dot{A})\quad \mathfrak{N}_\lambda:=(\mathfrak{M}_{\bar{\lambda}})^\perp. \tag{1}$$

The subspace $\mathfrak{N}_\lambda$ is called a *defect subspace* of $\dot{A}$ for the point $\bar{\lambda}$. The cardinal number $\dim\mathfrak{N}_\lambda$ remains constant when λ is in the upper half-plane. Similarly, the number $\dim\mathfrak{N}_\lambda$ remains constant when λ is in the lower half-plane. The numbers $\dim\mathfrak{N}_\lambda$ and $\dim\mathfrak{N}_{\bar\lambda}$ ($\mathrm{Im}\lambda < 0$) are called the *defect numbers* or *deficiency indices* of operator $\dot{A}$ [1]. The subspace $\mathfrak{N}_\lambda$ is the set of solutions of the equation $\dot{A}^*g = \lambda Pg$ and therefore can be written as $\mathfrak{N}_\lambda = \ker(\dot{A}^* - \lambda P)$. For further convenience, we will denote the deficiency subspaces of $\dot{A}$ by $\mathcal{N}_\pm$, that is,

$$\mathcal{N}_\pm = \ker(\dot{A}^* \mp Pi). \tag{2}$$

Let P_λ be the orthogonal projection onto $\mathfrak{N}_\lambda$, set

$$\mathfrak{B}_\lambda = P_\lambda\mathfrak{L}, \qquad \mathfrak{N}'_\lambda = \mathfrak{N}_\lambda \ominus \overline{\mathfrak{B}_\lambda}. \tag{3}$$

It is easy to see that $\mathfrak{N}'_\lambda = \mathfrak{N}_\lambda \cap \mathfrak{H}_0$ and $\mathfrak{N}'_\lambda$ is the set of solutions of the equation $\dot{A}^*g = \lambda g$ (see [21]), where $\dot{A}^* : \mathfrak{H} \to \mathfrak{H}_0$ is the adjoint operator to $\dot{A}$.

The subspace $\mathfrak{N}'_\lambda$ is the defect subspace of the densely defined symmetric operator $P\dot{A}$ on $\mathcal{H}_0$ ([21]). The numbers $\dim\mathfrak{N}'_\lambda$ and $\dim\mathfrak{N}'_{\bar\lambda}$ ($\mathrm{Im}\lambda < 0$) are called *semi-defect numbers* or the *semi-deficiency indices* of the operator $\dot{A}$ [13]. As in (2), we set

$$\mathcal{N}'_\pm = \ker(\dot{A}^* \mp i). \tag{4}$$

The von Neumann formula

$$\mathfrak{D}(\dot{A}^*) = \mathfrak{D}(\dot{A}) + \mathfrak{N}_\lambda + \mathfrak{N}_{\bar\lambda}, \qquad (\mathrm{Im}\lambda \neq 0), \tag{5}$$

for a non-densely defined operator $\dot{A}$ continue to hold, but this decomposition is not direct. There is another generalization of the von Neumann formulas [4], [13], [21] to the case of a non-densely defined symmetric operator.

From now on we will require our symmetric operator $\dot{A}$ to have equal deficiency indices, i.e., $\mathrm{def}(\dot{A}) = (n, n)$, $n \in \mathbb{N} \cup \{\infty\}$. It is known [13] that in this case $\dot{A}$ admits self-adjoint extensions A ($A^* = A$). Let A be such an extension. Then $\dot{A} \subset A$ and $PAx = \dot{A}^*x$ ($\forall x \in \mathfrak{D}(A)$). According to [13], an operator U ($\mathfrak{D}(U) \subseteq \mathfrak{N}_i$, $\mathfrak{R}(U) \subseteq \mathfrak{N}_{-i}$) is called an *admissible operator* if $(U - I)f_i \in \mathfrak{D}(\dot{A})$, $f_i \in \mathfrak{D}(U)$ implies that $f_i = 0$. Then (see [13]) any symmetric extension A of the non-densely defined closed symmetric operator $\dot{A}$, is defined by an isometric admissible operator U, with $\mathfrak{D}(U) \subseteq \mathcal{N}_+$, and $\mathfrak{R}(U) \subseteq \mathcal{N}_-$ by the formula

$$Af = \dot{A}f_{\dot{A}} + (-if_i - iUf_i), \quad f \in \mathfrak{D}(A),\ f_{\dot{A}} \in \mathfrak{D}(\dot{A}),\ f_i \in \mathcal{N}_+, \tag{6}$$

where

$$\mathfrak{D}(A) = \mathfrak{D}(\dot{A}) \dotplus (I - U)\mathfrak{D}(U). \tag{7}$$

The operator A is self-adjoint if and only if $\mathfrak{D}(U) = \mathcal{N}_+$ and $\mathfrak{R}(U) = \mathcal{N}_-$. The last two equations is a generalization of the von Neumann formula to the case of non-densely defined operator $\dot{A}$ obtained by M. Krasnoselskii [13].

We call two self-adjoint extensions A_1 and A_2 of $\dot{A}$ *relatively prime* if $\mathcal{D}(A_1) \cap \mathcal{D}(A_2) = \mathcal{D}(\dot{A})$. (In this case we shall also say that A_1 and A_2 are relatively prime w.r.t. $\dot{A}$). Let A_1 and A_2 be two distinct self-adjoint extensions of $\dot{A}$. The basic result on Krein's formula for the case of symmetric (densely defined) operator $\dot{A}$ with finite deficiency indices, as presented by Akhiezer and Glazman [1], then reads as follows.

THEOREM 2.1. (Krein's formula, [1])
There exists a $P_{1,2}(z) = (P_{1,2,j,k}(z))_{1\le j,k\le n} \in M_n(\mathbb{C})$, $z \in \rho(A_2) \cap \rho(A_1)$, *such that*

$$\det(P_{1,2}(z)) \neq 0, \;\; z \in \rho(A_2) \cap \rho(A_1), \tag{8}$$

$$P_{1,2}(z)^{-1} = P_{1,2}(z_0)^{-1} + (z - z_0)\big(u_{1,k}(z_0), u_{1,j}(\bar{z})\big), \;\; z, z_0 \in \rho(A_1), \tag{9}$$

$$(A_2 - z)^{-1} = (A_1 - z)^{-1} - \sum_{j,k=1}^{n} P_{1,2,j,k}(z)\big(\cdot\,, u_{1,k}(\bar{z})\big)u_{1,j}(z). \tag{10}$$

where $\{u_{1,j}(z)\}_{1\le j\le n}$ *is a basis for* $\mathfrak{N}_z$.

Reference [11] presents coefficients in the Krein formula for the case of a densely defined symmetric operator with deficiency $n \in \mathbb{N} \cup \{\infty\}$. The non-dense version for the coefficients in Krein's formula is given below in Theorem 4.1.

The literature on Krein's formula is very extensive. For a treatment of applications of the Krein's formula we refer to [2], [3], [18].

3. CAYLEY TRANSFORM AND THE FUNCTION $P_{1,2}(Z)$

For any self-adjoint extension A of $\dot{A}$ in $\mathcal{H}$ we introduce its unitary Cayley transform C_A by

$$C_A = (A + i)(A - i)^{-1}. \tag{11}$$

The following lemma is a modification of the similar result in [11] for the non-dense case.

LEMMA 3.1. *Let A, A_1, and A_2 be self-adjoint extensions of $\dot{A}$. Then*
(i). The Cayley transform of A maps $\mathcal{N}_-$ onto $\mathcal{N}_+$

$$C_A \mathcal{N}_- = \mathcal{N}_+. \tag{12}$$

(ii). $\mathcal{D}(A) = \mathcal{D}(\dot{A}) \dot{+} (I - C_A^{-1})\mathcal{N}_+$.
(iii). $\mathcal{N}_+$ *is an invariant subspace for* $C_{A_1}C_{A_2}^{-1}$ *and* $C_{A_2}C_{A_1}^{-1}$.
(iv). Suppose A_1 *and* A_2 *are relatively prime w.r.t.* $\dot{A}$. *Then*

$$\overline{\mathrm{ran}((A_2 - i)^{-1} - (A_1 - i)^{-1})} = \mathcal{N}_+, \tag{13}$$

$$\ker(((A_2 - i)^{-1} - (A_1 - i)^{-1})\big|_{\mathcal{N}_-}) = \{0\}. \tag{14}$$

Proof. A proof of (i), (iii), and (iv) can be replicated from the proof of the relevant lemma in [11] with some minor adjustments. Therefore we only prove (ii).

(ii). By (7),

$$\mathfrak{D}(A) = D(\dot{A}) \dot{+} (I - \mathcal{U}_A)\mathcal{N}_+. \tag{15}$$

for some admissible isometric operator $\mathcal{U}_A : \mathcal{N}_+ \to \mathcal{N}_-$. The inverse Cayley transform,

$$C_A^{-1} = (A - i)(A + i)^{-1},$$

is obviously an admissible operator in the above mentioned sense. Indeed, a direct check shows that $C_A^{-1} f = f$ implies $f = 0$ for any $f \in \mathcal{H}$. Also, since $I - C_A^{-1} = 2i(A + i)^{-1}$, $(I - C_A^{-1})\mathcal{N}_+ = 2i(A + i)^{-1}\mathcal{N}_+ \subseteq \mathcal{D}(A)$, one concludes

$$\mathcal{U}_A = C_A^{-1}\big|_{\mathcal{N}_+}. \tag{16}$$

∎

Next, assuming $A_\ell, \ell = 1, 2$ to be self-adjoint extensions of $\dot{A}$ and following [11], we define

$$P_{1,2}(z) = (A_1 - z)(A_1 - i)^{-1}((A_2 - z)^{-1} - (A_1 - z)^{-1})(A_1 - z)(A_1 + i)^{-1}, \tag{17}$$

$$z \in \rho(A_1) \cap \rho(A_2).$$

The following properties of $P_{1,2}(z)$ are needed.

Lemma 3.2. [11] *Let* $z, z' \in \rho(A_1) \cap \rho(A_2)$.
(i). $P_{1,2} : \rho(A_1) \cap \rho(A_2) \to \mathcal{B}(\mathcal{H})$ *is analytic and*

$$P_{1,2}(z)^* = P_{1,2}(\bar{z}). \tag{18}$$

(ii).

$$P_{1,2}(z)\big|_{\mathcal{N}_+^{\perp}} = 0, \quad P_{1,2}(z)\mathcal{N}_+ \subseteq \mathcal{N}_+. \tag{19}$$

(iii).

$$\begin{aligned} P_{1,2}(z) &= P_{1,2}(z') \\ &\quad + (z-z')P_{1,2}(z')(A_1+i)(A_1-z')^{-1}(A_1-i)(A_1-z)^{-1}P_{1,2}(z). \end{aligned} \tag{20}$$

(iv). $\operatorname{ran}(P_{1,2}(z)\big|_{\mathcal{N}_+})$ *is independent of* $z \in \rho(A_1) \cap \rho(A_2)$.

(v). Assume A_1 *and* A_2 *are relatively prime self-adjoint extensions of* $\dot{A}$. *Then* $P_{1,2}(z)\big|_{\mathcal{N}_+} : \mathcal{N}_+ \to \mathcal{N}_+$ *is invertible (i.e., one-to-one).*

(vi). Assume A_1 *and* A_2 *are relatively prime self-adjoint extensions of* $\dot{A}$. *Then*

$$\overline{\operatorname{ran}(P_{1,2}(i))} = \mathcal{N}_+. \tag{21}$$

(vii).

$$P_{1,2}(i)\big|_{\mathcal{N}_+} = (i/2)(I - C_{A_2}C_{A_1}^{-1})\big|_{\mathcal{N}_+}. \tag{22}$$

Next, let

$$C_{A_2}C_{A_1}^{-1}\big|_{\mathcal{N}_+} = -e^{-2i\alpha_{1,2}} \tag{23}$$

for some self-adjoint (possibly unbounded) operator $\alpha_{1,2}$ *in* $\mathcal{N}_+$. *If* A_1 *and* A_2 *are relatively prime, then*

$$\{(m+\tfrac{1}{2})\pi\}_{m\in\mathbb{Z}} \cap \sigma_p(\alpha_{1,2}) = \emptyset \tag{24}$$

and

$$(P_{1,2}(i)\big|_{\mathcal{N}_+})^{-1} = \tan(\alpha_{1,2}) - iI_{\mathcal{N}_+}. \tag{25}$$

In addition, $\tan(\alpha_{1,2}) \in \mathcal{B}(\mathcal{N}_+)$ *if and only if* $\operatorname{ran}(P_{1,2}(i)) = \mathcal{N}_+$.

Proof.
(i) is clear from (17).
(ii). Let $f \in \mathcal{D}(\dot{A})$, $g = (\dot{A}+i)f$. Then

$$P_{1,2}(z)g = (A_1-z)(A_1-i)^{-1}((A_2-z)^{-1} - (A_1-z)^{-1})(\dot{A}-z)f = 0 \tag{26}$$

yields $P_{1,2}(z)\big|_{\operatorname{ran}(\dot A+i)} = 0$ and hence $P_{1,2}(z)\big|_{\overline{\operatorname{ran}(\dot A+i)}} = P_{1,2}(z)\big|_{\mathcal{N}_+^\perp} = 0$ since $P_{1,2}(z) \in \mathcal{B}(\mathcal{H})$. Moreover, by (17)

$$\operatorname{ran}(P_{1,2}(z)) \subseteq (A_1 - z)(A_1 - i)^{-1} \ker(\dot A^* - Pz) \subseteq \ker(\dot A^* - Pi) = \mathcal{N}_+ \tag{27}$$

since

$$\begin{aligned}
&(\dot A^* - Pi)(A_1 - z)(A_1 - i)^{-1}\big|_{\ker(\dot A^* - Pz)} \\
&= (\dot A^* - Pi)(I - (z - i)(A_1 - i)^{-1})\big|_{\ker(\dot A^* - Pz)} \\
&= (\dot A^* - Pz + Pz - Pi - (z - i)(\dot A^* - Pi)(A_1 - i)^{-1})\big|_{\ker(\dot A^* - Pz)} \\
&= (P(z - i)I - (z - i)(P\dot A^* - Pi)(A_1 - i)^{-1})\big|_{\ker(\dot A^* - Pz)} = 0.
\end{aligned} \tag{28}$$

This proves (19).

Items (iii)– (vi) are proved in [11]. ▮

4. WEYL-TICHMARSH OPERATOR AND KREIN'S FORMULA

Here we define the Weyl-Tichmarsh operators associated with self-adjoint extensions of $\dot A$.

DEFINITION 4.1. Let A be a self-adjoint extension of $\dot A$, $\mathcal{N} \subseteq \mathcal{N}_+$ a closed linear subspace of $\mathcal{N}_+ = \ker(\dot A^* - Pi)$, and $z \in \rho(A)$. Then the Weyl-Tichmarsh operator $M_{A,\mathcal{N}}(z) \in \mathcal{B}(\mathcal{N})$ associated with the pair $(A, \mathcal{N})$ is defined by

$$M_{A,\mathcal{N}}(z) = P_{\mathcal{N}}(zA + I)(A - z)^{-1}\big|_{\mathcal{N}} = zI_{\mathcal{N}} + (1 + z^2)P_{\mathcal{N}}(A - z)^{-1}\big|_{\mathcal{N}}, \tag{29}$$

with $P_{\mathcal{N}}$ the orthogonal projection in $\mathcal{H}$ onto $\mathcal{N}$.

We need the following lemma and theorem that are modified versions of the corresponding results from [11].

LEMMA 4.1. [11] *Let A_ℓ, $\ell = 1, 2$ be relatively prime self-adjoint extensions of $\dot A$. Then*

$$(P_{1,2}(z)\big|_{\mathcal{N}_+})^{-1} = (P_{1,2}(i)\big|_{\mathcal{N}_+})^{-1} - (z - i)P_{\mathcal{N}_+}(A_1 + i)(A_1 - z)^{-1}P_{\mathcal{N}_+} \tag{30}$$

$$= \tan(\alpha_{1,2}) - M_{A_1,\mathcal{N}_+}(z), \;\; z \in \rho(A_1). \tag{31}$$

The theorem below gives Krein's formula in terms of Weyl-Tichmarsh operator-function.

THEOREM 4.1. (see [11]) *Let A_1 and A_2 be self-adjoint extensions of $\dot{A}$ and $z \in \rho(A_1) \cap \rho(A_2)$. Then*

$$(A_2 - z)^{-1} = (A_1 - z)^{-1} + (A_1 - i)(A_1 - z)^{-1} P_{1,2}(z)(A_1 + i)(A_1 - z)^{-1} \tag{32}$$

$$\begin{aligned} &= (A_1 - z)^{-1} + (A_1 - i)(A_1 - z)^{-1} P_{\mathcal{N}_{1,2,+}} \\ &\times (\tan(\alpha_{\mathcal{N}_{1,2,+}}) - M_{A_1,\mathcal{N}_{1,2,+}}(z))^{-1} P_{\mathcal{N}_{1,2,+}}(A_1 + i)(A_1 - z)^{-1}, \end{aligned} \tag{33}$$

where

$$\mathcal{N}_{1,2,+} = \ker((A_1\big|_{\mathcal{D}(A_1)\cap\mathcal{D}(A_2)})^* - P_1 i), \tag{34}$$

P_1 is orthoprojection onto $\overline{\mathcal{D}(A_1) \cap \mathcal{D}(A_2)}$, and

$$e^{-2i\alpha_{\mathcal{N}_{1,2,+}}} = -C_{A_2} C_{A_1}^{-1}\big|_{\mathcal{N}_{1,2,+}}. \tag{35}$$

Proof. If A_1 and A_2 are relatively prime w.r.t. $\dot{A}$, then $P_1 = P$ and Lemmas 3.1, 3.2, and 4.1 prove (32)–(35). If A_1 and A_2 are arbitrary self-adjoint extensions of $\dot{A}$ one replaces $\dot{A}$ by the largest common symmetric part of A_1 and A_2 given by $A_1\big|_{\mathcal{D}(A_1)\cap\mathcal{D}(A_2)}$. ∎

We should note that when operator $\dot{A}$ is densely defined, the orthoprojection operator P in (35) is identity. In this case the above theorem matches the corresponding result from [11].

COROLLARY 4.1.

$$P_{1,2}(i)\big|_{\mathcal{N}_{1,2,+}} = (i/2)(I - \mathcal{U}_{A_2}^{-1}\mathcal{U}_{A_1})\big|_{\mathcal{N}_{1,2,+}}, \tag{36}$$

where

$$\mathcal{U}_{A_\ell} = C_{A_\ell}^{-1}\big|_{\mathcal{N}_+},\ \ell = 1, 2 \tag{37}$$

denotes the linear isometric isomorphism from $\mathcal{N}_+$ onto $\mathcal{N}_-$ parametrizing the self-adjoint extensions A_ℓ of $\dot{A}$.

Proof. Combine (15), (16), and (22). ∎

Now we present the linear fractional transformation relating the Weyl-Tichmarsh operators $M_{A_\ell,\mathcal{N}_{1,2,+}}$ associated with two self-adjoint extensions A_ℓ, $(\ell = 1, 2)$, of $\dot{A}$. Even though the opeator $\dot{A}$ is not densely defined, the theorem below and its proof formally resembles the corresponding result in [11].

THEOREM 4.2. [11] *Suppose A_1 and A_2 are self-adjoint extensions of $\dot{A}$ and $z \in \rho(A_1) \cap \rho(A_2)$. Then*

$$\begin{aligned} M_{A_2,\mathcal{N}_+}(z) &= (P_{1,2}(i)\big|_{\mathcal{N}_+} + (I_{\mathcal{N}_+} + iP_{1,2}(i)\big|_{\mathcal{N}_+})M_{A_1,\mathcal{N}_+}(z)) \\ &\qquad \times ((I_{\mathcal{N}_+} + iP_{1,2}(i)\big|_{\mathcal{N}_+}) - P_{1,2}(i)\big|_{\mathcal{N}_+}M_{A_1,\mathcal{N}_+}(z))^{-1}, \qquad (38) \\ &= e^{-i\alpha_{1,2}}(\cos(\alpha_{1,2}) + \sin(\alpha_{1,2})M_{A_1,\mathcal{N}_+}(z)) \\ &\qquad \times (\sin(\alpha_{1,2}) - \cos(\alpha_{1,2})M_{A_1,\mathcal{N}_+}(z))^{-1}e^{i\alpha_{1,2}}, \qquad (39) \end{aligned}$$

where

$$e^{-2i\alpha_{1,2}} = -C_{A_2}C_{A_1}^{-1}\big|_{\mathcal{N}_+}, \tag{40}$$

$$P_{1,2}(i)\big|_{\mathcal{N}_+} = (i/2)(I - C_{A_2}C_{A_1}^{-1})\big|_{\mathcal{N}_+}, \tag{41}$$

$$I_{\mathcal{N}_+} + iP_{1,2}(i)\big|_{\mathcal{N}_+} = (1/2)(I + C_{A_2}C_{A_1}^{-1})\big|_{\mathcal{N}_+}. \tag{42}$$

5. RIGGED HILBERT SPACES AND ANALOGUES OF THE VON NEUMANN FORMULAS

In this section we are going to equip our Hilbert space $\mathcal{H}$ with spaces $\mathcal{H}_+$ and $\mathcal{H}_-$ called spaces with positive and negative norms, respectively [8]. We introduce a new Hilbert space $\mathcal{H}_+ = \mathfrak{D}(\dot{A}^*)$ $(\overline{(\mathfrak{D}(\dot{A}^*)} = \mathcal{H})$ with inner product

$$(f, g)_+ = (f, g) + (\dot{A}^* f, \dot{A}^* g) \qquad (f, g \in \mathcal{H}_+), \tag{43}$$

and then construct the *rigged* Hilbert space $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$. Here $\mathcal{H}_-$ is the space of all linear functionals over $\mathcal{H}_+$ that are continuous with respect to $\|\cdot\|_+$. The norms of these spaces are connected by the relations $\|x\| \le \|x\|_+$ $(x \in \mathcal{H}_+)$, and $\|x\|_- \le \|x\|$ $(x \in \mathcal{H})$. It is well known that there exists an isometric operator $\mathcal{R}$ which maps $\mathcal{H}_-$ onto $\mathcal{H}_+$ such that

$$\begin{aligned} (x, y)_- = (x, \mathcal{R}y) = (\mathcal{R}x, y) = (\mathcal{R}x, \mathcal{R}y)_+ \qquad (x, y \in \mathcal{H}_-), \\ (u, v)_+ = (u, \mathcal{R}^{-1}v) = (\mathcal{R}^{-1}u, v) = (\mathcal{R}^{-1}u, \mathcal{R}^{-1}v)_- \qquad (u, v \in \mathcal{H}_+). \end{aligned} \tag{44}$$

The operator $\mathcal{R}$ will be called the Riesz-Berezanskii operator. In what follows we use symbols $(+)$, $(\cdot)$, and $(-)$ to indicate the norms $\|\cdot\|_+$, $\|\cdot\|$, and $\|\cdot\|_-$ by which geometrical and topological concepts are defined in $\mathcal{H}_+$, $\mathcal{H}$, and $\mathcal{H}_-$.

We call an operator $\dot{A}$ *regular*, if $P\dot{A}$ is a closed operator in $\mathcal{H}_0$. Obviously any densely defined closed symmetric operator is regular. For a regular operator $\dot{A}$ we have

$$\mathcal{H}_+ = \mathfrak{D}(\dot{A}) + \mathfrak{N}'_\lambda + \mathfrak{N}'_{\bar{\lambda}} + \mathfrak{N}, \qquad (\text{Im}\lambda \neq 0) \tag{45}$$

where $\mathfrak{N} := \mathcal{R}\mathfrak{L}$. This is a generalization of von Neumann's formula. For $\lambda = \pm i$ we obtain the $(+)$-orthogonal decomposition

$$\mathcal{H}_+ = \mathfrak{D}(\dot{A}) \oplus \mathfrak{N}'_i \oplus \mathfrak{N}'_{-i} \oplus \mathfrak{N}. \tag{46}$$

By $P^+_{\mathfrak{M}}$ we denote the orthogonal projection in $\mathcal{H}_+$ onto $\mathfrak{M} = \mathfrak{N}'_i \oplus \mathfrak{N}'_{-i} \oplus \mathfrak{N}$. It was shown in [4] that operator $P^+_{\mathfrak{M}}$ maps the space $\mathfrak{N}_{\pm i}$ with $(\cdot)$-metrics bijectively and homeomorphically onto $\mathfrak{N} \oplus \mathfrak{N}'_{\pm i}$. Thus the inverse operator $\left(P^+_{\mathfrak{M}}\right)^{-1}$ exists.

Let $\tilde{A}$ be a closed symmetric extension of the operator $\dot{A}$. Then $\mathfrak{D}(\tilde{A}) \subset \mathcal{H}_+$ and $P\tilde{A}x = \dot{A}^*x$ $(\forall x \in \mathfrak{D}(\tilde{A}))$. Operator $\tilde{A}$ is regular if $P\tilde{A}$ is closed. According to [21], regularity of a closed symmetric extension $\tilde{A}$ is equivalent to $\mathfrak{D}(\tilde{A})$ being $(+)$-closed.

Let us now denote by $P^+_{\mathfrak{N}}$, the orthogonal projection operator from $\mathcal{H}_+$ onto $\mathfrak{N}$. We introduce a new inner product $(\cdot,\cdot)_1$ defined by

$$(f,g)_1 = (f,g)_+ + (P^+_{\mathfrak{N}} f, P^+_{\mathfrak{N}} g)_+ \tag{47}$$

for all $f, g \in \mathcal{H}_+$. The obvious inequality

$$\|f\|^2_+ \leq \|f\|^2_1 \leq 2\|f\|^2_+$$

shows that the norms $\|\cdot\|_+$ and $\|\cdot\|_1$ are topologically equivalent. It is easy to see that the spaces $\mathfrak{D}(A)$, $\mathfrak{N}'_i$, $\mathfrak{N}'_{-i}$, $\mathfrak{N}$ are (1)-orthogonal. We write $\mathfrak{M}_1$ for the Hilbert space $\mathfrak{M} = \mathfrak{N}'_i \oplus \mathfrak{N}'_{-i} \oplus \mathfrak{N}$ with inner product $(f,g)_1$. We denote by $\mathcal{H}_{+1}$ the space $\mathcal{H}_+$ with norm $\|\cdot\|_1$, and by $\mathcal{R}_1$ the corresponding Riesz-Berezanskii operator related to the rigged Hilbert space $\mathcal{H}_{+1} \subset \mathcal{H} \subset \mathcal{H}_{-1}$. The following theorem gives a characterization of the regular extensions for a regular closed symmetric operator $\dot{A}$ (see [4]).

Theorem 5.1. [4], [21] *I. For each closed symmetric extension $\tilde{A}$ of a regular operator $\dot{A}$ there exists a (1)-isometric operator $V = V(\tilde{A})$ on $\mathfrak{M}_1$ with the properties: a) $\mathfrak{D}(V)$ is (+)-closed and belongs to $\mathfrak{N} \oplus \mathfrak{N}'_i$, $\mathfrak{R}(V) \subset \mathfrak{N} \oplus \mathfrak{N}'_{-i}$; b) $Vh = h$ only for $h = 0$, and $\mathfrak{D}(\tilde{A}) = \mathfrak{D}(\dot{A}) \oplus (I+V)\mathfrak{D}(V)$.*

Conversely, for each (1)-isometric operator V with the properties a) and b) there exists a closed symmetric extension $\tilde{A}$ in the sense indicated.

II. The extension $\tilde{A}$ is regular if and only if the manifold $\mathfrak{R}(I+V)$ is (1)-closed.

III. The operator $\tilde{A}$ is self-adjoint if and only if $\mathfrak{D}(V) = \mathfrak{N}\oplus\mathfrak{N}'_i$, $\mathfrak{R}(V) = \mathfrak{N}\oplus\mathfrak{N}'_{-i}$.

An operator V that is described in the theorem 5.1 is related to operator U in (6)-(7) by the formula

$$VP^+_{\mathfrak{M}}g = P^+_{\mathfrak{M}}Ug, \qquad g\in\mathfrak{N}_i. \tag{48}$$

Now let A_1 and A_2 be relatively prime self-adjoint extensions of $\dot{A}$. According to theorem 5.1 there are two (1)-isometric operators $\mathcal{V}_{A_1}$ and $\mathcal{V}_{A_2}$ associated with operators A_1 and A_2 respectively. By (48) we have

$$\mathcal{V}_{A_1}P^+_{\mathfrak{M}}g = P^+_{\mathfrak{M}}\mathcal{U}_{A_1}g, \qquad \mathcal{V}_{A_2}P^+_{\mathfrak{M}}g = P^+_{\mathfrak{M}}\mathcal{U}_{A_2}g, \quad g\in\mathfrak{N}_i. \tag{49}$$

Consider operator $\mathcal{V}^{-1}_{A_2}\mathcal{V}_{A_1} : \mathfrak{N}\oplus\mathfrak{N}'_i \to \mathfrak{N}\oplus\mathfrak{N}'_i$. It is easy to see that this operator is unitary and

$$\mathcal{V}^{-1}_{A_2}\mathcal{V}_{A_1} = P^+_{\mathfrak{M}}\mathcal{U}^{-1}_{A_2}\mathcal{U}_{A_1}\left(P^+_{\mathfrak{M}}\right)^{-1}. \tag{50}$$

Consequently there exists a self-adjoint operator $\beta_{1,2}$ in $\mathfrak{N}\oplus\mathfrak{N}'_i$ such that

$$\mathcal{V}^{-1}_{A_2}\mathcal{V}_{A_1} = -e^{-2i\beta_{1,2}}. \tag{51}$$

Combining (40), (50), and (51) we get

$$e^{-2i\alpha_{1,2}} = \left(P^+_{\mathfrak{M}}\right)^{-1} e^{-2i\beta_{1,2}} P^+_{\mathfrak{M}}. \tag{52}$$

Similar relationships can be obtained between sin, cos, and tan of operators $\alpha_{1,2}$ and $\beta_{1,2}$, respectively. These relationships will allow us to relate Krein's formula and analogues of the von Neumann parametrization (46) for the non-dense case. In particular we have

$$P_{1,2}(i)\big|_{\mathcal{N}_{1,2,+}} = (i/2)\left(I - \left(P^+_{\mathfrak{M}}\right)^{-1}\mathcal{V}^{-1}_{A_2}\mathcal{V}_{A_1}P^+_{\mathfrak{M}}\right)\big|_{\mathcal{N}_{1,2,+}}, \tag{53}$$

and theorem 4.1 can be re-written in the following form

THEOREM 5.2. *Let A_1 and A_2 be self-adjoint extensions of $\dot{A}$ and $z\in\rho(A_1)\cap\rho(A_2)$. Then*

$$\begin{aligned}(A_2-z)^{-1} &= (A_1-z)^{-1} + (A_1-i)(A_1-z)^{-1}P_{\mathcal{N}_{1,2,+}}\\ &\times \left(\left(P^+_{\mathfrak{M}}\right)^{-1}\tan(\beta_{\mathcal{N}_{1,2,+}})P^+_{\mathfrak{M}} - M_{A_1,\mathcal{N}_{1,2,+}}(z)\right)^{-1}P_{\mathcal{N}_{1,2,+}}(A_1+i)(A_1-z)^{-1},\end{aligned} \tag{54}$$

where

$$\mathcal{N}_{1,2,+} = \ker((A_1\big|_{\mathcal{D}(A_1)\cap\mathcal{D}(A_2)})^* - P_1 i), \tag{55}$$

P_1 *is orthoprojection onto* $\overline{\mathcal{D}(A_1)\cap\mathcal{D}(A_2)}$*, and*

$$e^{-2i\beta_{\mathcal{N}_{1,2,+}}} = -\mathcal{V}_{A_2}^{-1}\mathcal{V}_{A_1}\big|_{\mathcal{N}_{1,2,+}}. \tag{56}$$

The theorem describing the relationship between functions $M_{A_1}(z)$ and $M_{A_2}(z)$ can be given now in terms of the operator $\beta_{1,2}$. In particular, theorem 4.2 now takes the following form:

THEOREM 5.3. *Suppose A_1 and A_2 are self-adjoint extensions of $\dot{A}$ and $z\in\rho(A_1)\cap\rho(A_2)$. Then*

$$\begin{aligned} M_{A_2,\mathcal{N}_+}(z) = \left(P_{\mathfrak{M}}^+\right)^{-1} e^{-i\beta_{1,2}}\left[\cos(\beta_{1,2})P_{\mathfrak{M}}^+ + \sin(\beta_{1,2})P_{\mathfrak{M}}^+ M_{A_1,\mathcal{N}_+}(z)\right] \\ \times\left[\sin(\beta_{1,2})P_{\mathfrak{M}}^+ - \cos(\beta_{1,2})P_{\mathfrak{M}}^+ M_{A_1,\mathcal{N}_+}(z)\right]^{-1} e^{i\beta_{1,2}}P_{\mathfrak{M}}^+, \end{aligned} \tag{57}$$

*where $e^{-2i\beta_{1,2}}$ is defined by (*56*).*

6. EXAMPLES

EXAMPLE 6.1. Let $\mathcal{H}=\mathbb{C}^3$. We define operator $\dot{A}$ as follows

$$\dot{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} \tag{58}$$

with $\mathfrak{D}(\dot{A}) = \{\vec{x}\in\mathbb{C}^3 \mid \vec{x}=(x_1,x_2,0)\}$. If we consider operator $\dot{A}$ as acting from $\mathbb{C}^2$ into $\mathbb{C}^3$ then its adjoint is

$$\dot{A}^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix} \tag{59}$$

defined on entire $\mathbb{C}^3$. It is easy to see that $(\dot{A}x,y)=(x,\dot{A}^*y)$ for all $x\in\mathfrak{D}(\dot{A})$ and $y\in\mathbb{C}^3$. An orthoprojection $P:\mathbb{C}^3\to\mathbb{C}^2$ then takes the form

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{60}$$

Next, we introduce two self-adjoint extensions of $\dot{A}$

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & a \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & b \end{pmatrix}, \tag{61}$$

where a and b are two real numbers and $a > b$. One can verify that A_1 and A_2 are relatively prime iff $a \neq b$. Deficiency spaces $\mathcal{N}_\pm = \ker(\dot{A}^* \mp Pi)$ are one dimensional subspaces of $\mathbb{C}^3$

$$\mathcal{N}_+ = \{\vec{z} \in \mathbb{C}^3 \mid \vec{z} = (0, z, z)\} \tag{62}$$

and

$$\mathcal{N}_- = \{\vec{z} \in \mathbb{C}^3 \mid \vec{z} = (0, -z, z)\}, \tag{63}$$

respectively. Hence $\operatorname{def}(\dot{A}) = (1,1)$. It is easy to see now that semi-deficiency subspaces of $\dot{A}$ are trivial. Therefore if $\mathcal{H}$ is equipped as a rigged triplet the projection operator $P^+_{\mathfrak{M}}$ acts as the identity mapping on $\mathfrak{N}$ making (52) trivial.

The Cayley transform C_{A_1} of the self-adjoint extension A_1, for example, is

$$C_{A_1} = (A+i)(A-i)^{-1} = \frac{1}{a-2i} \begin{pmatrix} 2+ai & 0 & 0 \\ 0 & -a & 2i \\ 0 & -2i & a \end{pmatrix} \tag{64}$$

Its inverse is given by

$$C_{A_1}^{-1} = \frac{1}{a+2i} \begin{pmatrix} 2-ai & 0 & 0 \\ 0 & -a & 2i \\ 0 & -2i & a \end{pmatrix}.$$

A direct check shows that $C_{A_1} : \mathcal{N}_- = \mathcal{N}_+$ and that $C_{A_2} C_{A_1}^{-1} : \mathcal{N}_+ = \mathcal{N}_+$, where

$$C_{A_2} C_{A_1}^{-1} = \frac{1}{(a+2i)(b-2i)} \begin{pmatrix} (2-ai)(2+bi) & 0 & 0 \\ 0 & 4+ab & 2(a-b)i \\ 0 & 2(a-b)i & 4+ab \end{pmatrix}. \tag{65}$$

Now we can evaluate $-e^{-2i\beta_{1,2}} = C_{A_2} C_{A_1}^{-1}\big|_{\mathcal{N}_+}$. Simple calculations show that for any $z \in \mathcal{N}_+$

$$-e^{-2i\beta_{1,2}} z = \chi z,$$

where

$$\chi = \frac{(4+a\,b) - 2\,(a-b)i}{(4+ab)+2(a-b)i}. \tag{66}$$

It is easy to see now that

$$1 \in \sigma_p(C_{A_1}C_{A_2}^{-1}|_{\mathcal{N}_+}),$$

iff $\chi = 1$ which can happen only when $a = b$. This confirms that A_1 and A_2 are relatively prime iff $a \neq b$.

Furthermore,

$$\tan\beta_{1,2} = \frac{\chi+1}{\chi-1}i = \frac{4+ab}{2(a-b)}. \tag{67}$$

The left-hand side of the Krein formula reads

$$(A_2 - zI)^{-1} - (A_1 - zI)^{-1} = \frac{a-b}{(z^2-bz-1)(z^2-az-1)}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & zi \\ 0 & -zi & z^2 \end{pmatrix}. \tag{68}$$

In particular,

$$(A_2 - iI)^{-1} - (A_1 - iI)^{-1} = \frac{a-b}{(2+ai)(2+bi)}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \tag{69}$$

which immediately confirms (13). Function $P_{1,2}(z)$ now has a form

$$P_{1,2}(z) = \frac{(a-b)(z^2-az-1)}{(4+a^2)(z^2-bz-1)}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \tag{70}$$

where

$$P_{1,2}(i) = \frac{(a-b)}{(2-ai)(2+bi)}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}. \tag{71}$$

For any $\vec{z} \in \mathcal{N}_+$,

$$P_{1,2}(i)\vec{z} = 2\frac{(a-b)}{(2-ai)(2+bi)}\vec{z},$$

and one easily checks that

$$\left(P_{1,2}(i)\big|_{\mathcal{N}_+}\right)^{-1} = (\tan\beta_{1,2} - iI)\,\big|_{\mathcal{N}_+} = \frac{(2-ai)(2+bi)}{2(a-b)}. \tag{72}$$

The functions $M_{A_1,\mathcal{N}_+}(z)$ and $M_{A_2,\mathcal{N}_+}(z)$ are

$$M_{A_1,\mathcal{N}_+}(z) = \frac{az^2+4z-a}{-4z^2+4az+4}\begin{pmatrix}0&0&0\\0&1&1\\0&1&1\end{pmatrix}, \tag{73}$$

and

$$M_{A_2,\mathcal{N}_+}(z) = \frac{bz^2+4z-b}{-4z^2+4bz+4}\begin{pmatrix}0&0&0\\0&1&1\\0&1&1\end{pmatrix}. \tag{74}$$

In order to verify (38) we notice that

$$\sin\beta_{1,2} = \frac{4+ab}{\sqrt{a^2+4}\sqrt{b^2+4}}, \qquad \cos\beta_{1,2} = \frac{2(a-b)}{\sqrt{a^2+4}\sqrt{b^2+4}}. \tag{75}$$

and the right-hand side of (57) exactly matches (74).

Example 6.2. Let $\mathcal{H} = L^2((0,\infty);dx)$ and

$$-\nabla^2 = -\frac{d^2}{dx^2}. \tag{76}$$

Let

$$\mathcal{D} = \big\{g \in L^2_{((0,\infty);dx)} \,|\, g, g' \in AC_{\rm loc}((0,\infty)),\ g(x)\equiv 0, \forall x\in(0,1], \\ g'(1+) = 0\big\}, \tag{77}$$

and

$$\mathcal{H}_0 = \overline{\mathcal{D}} = \{g\in L^2((0,\infty);dx)\,\big|\, g(x)\equiv 0,\ \forall x\in(0,1]\}. \tag{78}$$

It is easy to verify that

$$\mathfrak{B} := \mathcal{H}\ominus\mathcal{H}_0 = \{g\in L^2((0,\infty);dx)\,|\, g(x) \overset{\text{a.e.}}{\equiv} 0,\ \forall x\in[1,+\infty)\}. \tag{79}$$

We define an operator $\dot{A} : \mathcal{H}_0 \to \mathcal{H} = \mathcal{H}_0 \oplus \mathfrak{B}$ on $\mathcal{D}(\dot{A})$ as

$$\dot{A} = \begin{bmatrix}-\nabla^2 & 0\\ 0 & 0\end{bmatrix},\ \text{where } \mathcal{D}(\dot{A}) = \left\{\begin{bmatrix} g\\0\end{bmatrix}\,\Big|\, g\in\mathcal{D}\right\}. \tag{80}$$

Then $\dot{A}^*:\mathcal{H}_0\oplus\mathfrak{B}\to\mathcal{H}_0$ is given by

$$\dot{A}^*=\begin{bmatrix}-\nabla^2 & 0\\ 0 & 0\end{bmatrix} \tag{81}$$

with

$$\mathcal{D}(\dot{A}^*)=\left\{\begin{bmatrix}g_0\\ g_b\end{bmatrix}\ \Big|\ g_0\in\mathcal{H}_0, g_b\in\mathfrak{B}, g_0, g_0'\in AC_{\mathrm{loc}}(1,\infty),\right.$$
$$\left. g_b, g_b'\in AC_{\mathrm{loc}}(0,1)\right\}. \tag{82}$$

A projection operator $P:\mathcal{H}=\mathcal{H}_0\oplus\mathfrak{B}\to\mathcal{H}_0$ is defined as follows

$$P=\begin{bmatrix}I_{\mathcal{H}_0} & 0\\ 0 & 0\end{bmatrix}. \tag{83}$$

Now one can verify that the semideficiency subspace

$$\mathcal{N}'_z=\ker(\dot{A}^*-z),$$

is the set of functions $g(x)\in\mathcal{D}(\dot{A}^*)$ such that

$$g(x)=\begin{bmatrix}ce^{i\sqrt{z}x}\\ 0\end{bmatrix}, c\in\mathbb{C}. \tag{84}$$

Similarly the deficiency space

$$\mathcal{N}_z=\ker(\dot{A}^*-Pz)$$

contains all the functions $g(x)=g_0(x)+g_b(x)$, $(g_0(x)\in\mathcal{H}_0,\ g_b(x)\in\mathfrak{B})$ from $\mathcal{D}(\dot{A}^*)$ such that

$$g(x)=\begin{bmatrix}ce^{i\sqrt{z}x}\\ g_b(x)\end{bmatrix}, c\in\mathbb{C}. \tag{85}$$

So we conclude that $\mathrm{def}(\dot{A})=(\infty,\infty)$.

Now let A be an arbitrary self-adjoint extension of $\dot{A}$. The formula for the resolvent of A can be found as a solution of the differential equation

$$(A-\lambda I)y(x)=f(x)$$

that belongs to $L^2((0,\infty);dx)$. If $f(x)=f_0(x)+f_b(x)$, $(f_0(x)\in\mathcal{H}_0$, $f_b(x)\in\mathfrak{B})$ then

$$y(x) = (A - \lambda I)^{-1} f(x) = (A - \lambda I)^{-1} \begin{bmatrix} f_0(x) \\ f_b(x) \end{bmatrix} \tag{86}$$

$$= \begin{bmatrix} C_1 e^{i\sqrt{\lambda}x} + \frac{i}{2\sqrt{\lambda}}\,\Phi_0 \\ \\ C_2 e^{i\sqrt{\lambda}x} + C_3 e^{-i\sqrt{\lambda}x} + \frac{i}{2\sqrt{\lambda}}\,\Phi_{\mathfrak{B}} \end{bmatrix} \tag{87}$$

where

$$\Phi_0 = e^{i\sqrt{\lambda}x} \int_1^x f_0(t) e^{-i\sqrt{\lambda}t}\, dt + e^{-i\sqrt{\lambda}x} \int_x^\infty f_0(t) e^{i\sqrt{\lambda}t}\, dt,$$

and

$$\Phi_{\mathfrak{B}} = e^{i\sqrt{\lambda}x} \int_0^x f_b(t) e^{-i\sqrt{\lambda}t}\, dt + e^{-i\sqrt{\lambda}x} \int_x^1 f_b(t) e^{i\sqrt{\lambda}t}\, dt.$$

We should note that the values of the constants $\{C_1, C_2, C_3\}$ depend on the function $f(t)$ and λ. Now we introduce two self-adjoint extensions A_1 and A_2 of $\dot{A}$

$$A_1 = A_2 = \begin{bmatrix} -\nabla^2 & 0 \\ 0 & -\nabla^2 \end{bmatrix},$$

with

$$\mathcal{D}(A_1) = \left\{ \begin{bmatrix} g_0 \\ g_b \end{bmatrix} \in \mathcal{D}(\dot{A}^*) \middle| \; g_b(0_+) = 0,\; g_b(1_-) = g_b'(1_-),\; g_0(1_+) = 0 \right\}, \tag{88}$$

$$\mathcal{D}(A_2) = \left\{ \begin{bmatrix} g_0 \\ g_b \end{bmatrix} \in \mathcal{D}(\dot{A}^*) \middle| \; g_b(0_+) = 0, g_b(1_-) = g_b'(1_-),\; g_0(1_+) = g_0'(1_+) \right\} \tag{89}$$

These extensions are not relatively prime. One can see that their maximal symmetric part is wider than $\dot{A}$ and is a symmetric densely defined operator in $\mathcal{H}$. Using the initial conditions one can find a set of constants

$\{C_1,\, C_2,\, C_3\}$ for each extension. Straightforward calculations then yield

$$\begin{aligned}\big((A_2 - zI)^{-1} - (A_1 - zI)^{-1}\big) f(x) &= \frac{ie^{-2i\sqrt{z}}}{i+\sqrt{z}}\, P \int_1^{\infty} f_0(x)\, e^{i\sqrt{z}x}\, dx \cdot e^{i\sqrt{z}x} \\ &= \frac{ie^{-2i\sqrt{z}}}{i+\sqrt{z}}\, P \left(f_0(x),\, \overline{e^{i\sqrt{z}x}} \right) e^{i\sqrt{z}x}.\end{aligned} \tag{90}$$

We should mention that the self-adjoint extensions A_1 and A_2 are selected such that the function $P_{12}(z)$ is zero on entire subspace $\mathfrak{B}$. Similarly, for $f(x) = f_0(x) + f_b(x)$

$$P_{12}(z) f(x) = P_{12}(z)(f_0(x) + f_b(x)) = P_{12}(z) f_0(x).$$

Let us write $\mathcal{N}_+$ as follows

$$\mathcal{N}_+ = \mathcal{N}_+' \oplus \mathcal{N}_+^{\mathfrak{B}},$$

where $\mathcal{N}_+^{\mathfrak{B}}$ is an orthogonal complement to $\mathcal{N}_+'$ in $\mathcal{N}_+$. It can be seen that both $\mathcal{N}_+'$ and $\mathcal{N}_+^{\mathfrak{B}}$ are invariant w.r.t. the Weyl-Tichmarsh functions $M_{A_1,\mathcal{N}_+}(z)$ and $M_{A_2,\mathcal{N}_+}(z)$. Consequently we write

$$M_{A_k,\mathcal{N}_+}(z) = \begin{bmatrix} M_{A_k,\mathcal{N}_+'}(z) & 0 \\ 0 & M_{A_k,\mathcal{N}_+^{\mathfrak{B}}}(z) \end{bmatrix}, \quad (k = 1, 2) \tag{91}$$

The main goal of this example is to illustrate Theorem 4.2. We note that the self-adjoint extensions A_1 and A_2 are constructed in a such a way that $M_{A_1,\mathcal{N}_+^{\mathfrak{B}}}(z) = M_{A_2,\mathcal{N}_+^{\mathfrak{B}}}(z)$. Consequently, $P_{1,2}(i)\big|_{\mathcal{N}_+^{\mathfrak{B}}} = 0$ and formula (38) trivially holds for an arbitrary $g_b(x) \in \mathcal{N}_+^{\mathfrak{B}}$. Thus we need to concentrate only on $\mathcal{H}_0$ component of $\mathcal{N}_+$ which is in fact $\mathcal{N}_+'$.

In order to proceed we are going to use (86) to derive the formula for $C_A\big|_{\mathcal{N}_-'} : \mathcal{N}_-' \to \mathcal{N}_+'$, where

$$C_A = (A + i)(A - i)^{-1},$$

is a Cayley transform of a self-adjoint extension of operator $\dot{A}$. This yields

$$C_A[e^{i\sqrt{-i}x}] = \left(2iC_1 + \sqrt{\frac{i}{2}}\, e^{-\sqrt{2}i} \right) e^{i\sqrt{i}x}, \tag{92}$$

where C_1 is defined by (86)-(89). Therefore, to compute

$$C_{A_2}C_{A_1}^{-1}\Big|_{\mathcal{N}'_+} = \mathcal{U}_2^{-1}\mathcal{U}_1\Big|_{\mathcal{N}'_+} = -e^{-2i\alpha_{1,2}}\Big|_{\mathcal{N}'_+}$$

we use (92) with appropriate values for C_1 determined by the initial conditions in (88)-(89) for each of the extensions A_1 and A_2. Consequently

$$C_{A_2}C_{A_1}^{-1}[e^{i\sqrt{i}x}] = \frac{1+i}{\sqrt{2}}\, e^{i\sqrt{i}x}, \tag{93}$$

and $\alpha_{1,2} = 3\pi/8$. Performing straightforward though tedious calculations we find that

$$P_{12}(z)\Big|_{\mathcal{N}'_+} = \frac{i}{\sqrt{2}(\sqrt{z}+i)}, \tag{94}$$

and

$$P_{12}(i)\Big|_{\mathcal{N}'_+} = \frac{1+(\sqrt{2}-1)i}{2\sqrt{2}}. \tag{95}$$

Computing the two functions $M_{A_1,\mathcal{N}'_+}(z)$ and $M_{A_2,\mathcal{N}'_+}(z)$ using the definition (29), we get

$$M_{A_1,\mathcal{N}'_+}(z) = 1 + \sqrt{2z}\, i, \tag{96}$$

and

$$M_{A_2,\mathcal{N}'_+}(z) = \frac{(\sqrt{2}+1)(i-\sqrt{z})}{(\sqrt{z}+i)}. \tag{97}$$

Now one can easily verify the result of Theorem 4.2.

ACKNOWLEDGMENTS

We would like to thank Fritz Gesztesy and Konstantin Makarov for valuable discussions.

REFERENCES

1. N. I. Akhiezer and I. M. Glazman. "Theory of Linear Operators in Hilbert Space", Dover, New York, 1993.

2. S. Albeverio, F. Gesztesy, R. Høegh-Krohn, H. Holden, "Solvable Models in Quantum Mechanics", Springer, Berlin, 1988.
3. S. Albeverio, P. Kurasov. "Singular Pertrubations of Differential Operators", *London Math. Soc. Lecture Notes* **271**, Cambridge Univesity Press, London, 2000.
4. Yu.M.Arlinskiĭ, E.R.Tsekanovskiĭ, The method of equipped spaces in the theory of extensions of Hermitian operators with a nondense domain of definition, *Sibirsk. Mat. Zh.* **15**, (1974), 597–610.
5. N. Aronszajn, On a problem of Weyl in the theory of singular Sturm-Liouville equations, *Amer. J. Math.* **79**, (1957), 597–610.
6. S.V. Belyi, E.R. Tsekanovskiĭ. Realization theorems for operator-valued R-functions, *Operator theory: Advances and Applications*, **98**, Birkhäuser Verlag Basel, (1997), 55–91.
7. S.V. Belyi, E.R. Tsekanovskiĭ. On classes of realizable operator-valued R-functions, *Operator theory: Advances and Applications*, **115**, Birkhäuser Verlag Basel, (2000), 85–112.
8. Ju.M. Berezanskii. "Expansion in eigenfunctions of self-adjoint operators", vol. 17, Transl. Math. Monographs, AMS, Providence, 1968.
9. W. F. Donoghue, On the perturbation of spectra, *Commun. Pure Appl. Math.* **18**, (1965), 559–579.
10. F. Gesztesy, E.R. Tsekanovskiĭ, On matrix-valued Herglotz functions, *Math. Nach.* **218**, (2000), 61–138.
11. F. Gesztesy, K.A. Makarov, E. Tsekanovskiĭ, An addendum to Krein's formula, *J. Math. Anal. Appl.* **222**, (1998), 594–606.
12. F. Gesztesy, N.J. Kalton, K.A. Makarov, E. Tsekanovskiĭ, Some Applications of Operator-Valued Herglotz Functions, *Operator Theory: Advances and Applications*, **123**, Birkhäuser, Basel, (2001), 271–321.
13. M.A. Krasnoselskii, On self-adjoint extensions of Hermitian operators, *Ukrain. Mat. Zh.* **1**, (1949), 21–38.
14. M. G. Krein, On Hermitian operators with deficiency indices one, *Dokl. Akad. Nauk SSSR*, **43**, (1944), 339–342, (Russian).
15. M. G. Krein, Resolvents of a Hermitian operator with defect index (m, m), *Dokl. Akad. Nauk SSSR*, **52**, (1946), 657–660, (Russian).
16. M. M.Malamud, On a formula of the generalized resolvents of a nondensely defined Hermitian operator, *Ukrain. Math. J.* **44**, (1992), 1522–1547.
17. J. von Neumann, Allgemeine Eigenwerttheorie hermitescher Funktionaloperatoren, *Math. Ann.* **102**, (1929-30), 49–131.
18. B.S. Pavlov, Operator Extensions Theory and explicitly solvable models, *Uspekhi Math. Nauk*, **42**, (1987), 99–131.
19. Sh. N. Saakjan, On the theory of the resolvents of a symmetric operator with infinite deficiency indices, *Dokl. Akad. Nauk Arm. SSR*, **44**, (1965), 193–198, (Russian).
20. Ju.L. Šmuljan, Extended resolvents and extended spectral functions of Hermitian operator, *Math. USSR Sbornick*, **13**, no. 3, (1971), 435–450.
21. E. R. Tsekanovskiĭ, Ju.L. Šmuljan, The theory of bi-extensions of operators on rigged Hilbert spaces. Unbounded operator colligations and characteristic functions, *Russ. Math. Surv.*, **32**, (1977), 73–131.
22. E. R. Tsekanovskiĭ, Ju.L. Šmuljan, "Method of generalized functions in the theory of extensions of unbounded linear operators", Donetsk State Univeristy, Donetsk, 1973.