Complex Analysis and Operator Theory (2022) 16:107
https://doi.org/10.1007/s11785-022-01288-7

Complex Analysis and Operator Theory

On the c-Entropy of L-Systems with Schrödinger Operator

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Received: 31 January 2022 / Accepted: 28 September 2022

Abstract

We study L-systems whose main operators are extensions of one-dimensional half-line Schrödinger operators with deficiency indices $(1, 1)$, the Schrödinger L-systems. Introducing new concepts of an c-entropy and dissipation coefficient for an L-system we discuss the following dual problems: describe Schrödinger L-systems (1) with a given c-entropy and minimal dissipation coefficient, and (2) with a given dissipation coefficient and maximal c-entropy. Also, we analyze in detail the dual c-entropy problems for Schrödinger L-systems with sectorial and extremal main operators.

Keywords L-system · Transfer function · Impedance function · Herglotz–Nevanlinna function · Donoghue class · Sectorial operator · Extremal operator · Schrödinger operator

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In loving memory of Moshe Livšic

Communicated by Fabrizio Colombo.

This article is part of the topical collection "Spectral Theory and Operators in Mathematical Physics" edited by Jussi Behrndt and Fabrizio Colombo.

This paper is dedicated to the memory of Moshe Livšic, a remarkable Human Being and Mathematician. His pioneering research in the theory of non-selfadjoint operators and system theory [16] has made writing this paper possible. The following lines from a Russian poet Nikolaj Gumilev (Prophets, 1905) describe the life and the light of scientific accomplishments of Livšic really well

И ныне есть еще пророки,
Хотя упали алтари,
Их очи ясны и глубоки,
Грядущим пламенем зари.

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Published online: 15 October 2022

Mathematics Subject Classification Primary 47A10; Secondary 47N50 · 81Q10

1 Introduction

In the current paper we set focus on L-systems whose main operators are extensions of the minimal one-dimensional Schrödinger operators with deficiency indices (1, 1) on the half-line. We refer to such L-systems as *Schrödinger L-systems*. As a main highlight of the current paper we introduce a new concept of an L-system *coupling entropy* (or *c-entropy*), which is an additive quantity with respect to the coupling of L-systems (see [7] for the concept of coupling). We relate the c-entropy of a Schrödinger L-system to the dissipation coefficient of its main operator T_h and pose the following *dual c-entropy problems*. The first problem is to describe a Schrödinger L-system with a given c-entropy and minimal dissipation coefficient, while the second one is to construct a Schrödinger L-system that has a given dissipation coefficient and maximal c-entropy. We solve these dual problems for several classes of Schrödinger L-systems, in particular, for the Schrödinger L-systems whose impedance functions belong to one of the generalized Donoghue classes $\mathfrak{M}_\kappa$ and $\mathfrak{M}_\kappa^{-1}$ introduced in [6]. As an auxiliary result, we obtain a criteria (in terms of the boundary value of the main operators) for the impedance functions to belong to one of (generalized) Donoghue classes $\mathfrak{M}$, $\mathfrak{M}_\kappa$, and $\mathfrak{M}_\kappa^{-1}$ and then solve the dual c-entropy problems for Schrödinger L-systems with extremal and β-sectorial main operators.

The paper is organized as follows. The formal definitions of general and Schrödinger L-systems as well as corresponding function classes are presented in Sects. 2, 3 and 4. We capitalize on the fact that class of Schrödinger L-systems $\Theta_{\mu,h}$ forms a two-parametric family whose members are uniquely defined by a real-valued parameter μ and a complex boundary value h, ($\operatorname{Im} h > 0$) of the main operator. Here the parameter $\mu \in \mathbb{R} \cup \{\infty\}$ uniquely defines (for a given h) the state-space operator $\mathbb{A}_{\mu,h}$ of the L-system $\Theta_{\mu,h}$, thus fixing $\Theta_{\mu,h}$ in a unique way.

In Sect. 4 we establish a connection between the absolute value of the von Neumann parameter of the main operator T_h and its boundary value parameter h and put forward the definition of the dissipation coefficient. Also, we obtain a criteria for the impedance functions of L-systems with Schrödinger operator to fall into one of the (generalized) Donoghue classes $\mathfrak{M}$, $\mathfrak{M}_\kappa$, and $\mathfrak{M}_\kappa^{-1}$ (see Theorems 8-10). It is worth mentioning that Theorem 8 describes an entire family of Schrödinger L-systems whose impedance functions belong to the Donoghue class $\mathfrak{M}$, while Theorems 9-10 deal with explicitly defined Schrödinger L-systems whose impedance functions are members of the generalized Donoghue classes $\mathfrak{M}_\kappa$ and $\mathfrak{M}_\kappa^{-1}$, respectively.

In Sects. 5 and 6 we discuss an analogue of the Phillips-Kato extension problem for a non-negative Schrödinger operator with deficiency indices (1, 1) in the Hilbert space $\mathcal{H} = L^2(\ell, \infty)$. Recall that in general the Phillips-Kato extension problem concerns the existence and description of all maximal accretive and/or sectorial non-self-adjoint extensions A of a non-negative symmetric operator $\dot{A}$ such that $\dot{A} \subset A \subset \dot{A}^*$.

In Sect. 7 we introduce the concept of an L-system c-entropy and relate it to L-system's dissipation coefficient introduced in Sect. 4. Two following dual problems associated with the c-entropy of Schrödinger L-systems are considered:

- *Give a description of an L-system with the Schrödinger dissipative main operator that has a given c-entropy and the minimal dissipation coefficient.*
- *Describe an L-system with Schrödinger main operator that has a given dissipation coefficient and the maximal c-entropy.*

In Sects. 7 and 8 we respectively present the solutions to both problems posed for the Schrödinger L-systems whose impedance functions belong to one of the generalized Donoghue classes $\mathfrak{M}_\kappa$ and $\mathfrak{M}_\kappa^{-1}$.

In Sect. 9 we solve the dual c-entropy problems for the classes of Schrödinger L-systems with extremal and β-sectorial main operators. Dealing with these two cases we do not require that the impedance functions of the Schrödinger L-systems in question belong to one of the generalized Donoghue classes. As a result, we present the solution to dual c-entropy problems for the entire one-parametric family of Schrödinger L-systems $\Theta_{\mu,h}$. We also treat a combined case of extremal and β-sectorial main operators to see when the corresponding L-system has a maximal c-entropy. It turns out that the Schrödinger L-system with accretive (either β-sectorial ($\beta \in (0, \pi/2)$ or extremal) main operator T_h attains the maximum c-entropy when the main operator is extremal accretive. Moreover, in the case when both main and state-space operators are extremal, the quasi-kernel in the corresponding Schrödinger L-system coincides with the Krein-von Neumann extension of the underlying symmetric operator. In addition, we find the conditions when the impedance function of the Schrödinger L-system with the main extremal operator that has a maximum c-entropy belongs to the generalized Donoghue classes $\mathfrak{M}_\kappa$ or $\mathfrak{M}_\kappa^{-1}$, respectively.

We conclude the paper with providing examples that illustrate main results.

2 Preliminaries

For a pair of Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ we denote by $[\mathcal{H}_1, \mathcal{H}_2]$ the set of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. Let $\dot{A}$ be a closed, densely defined, symmetric operator in a Hilbert space $\mathcal{H}$ with inner product (f, g), $f, g \in \mathcal{H}$. Any non-symmetric operator T in $\mathcal{H}$ such that

$$\dot{A} \subset T \subset \dot{A}^*$$

is called a *quasi-self-adjoint extension* of $\dot{A}$.

Consider the rigged Hilbert space (see [2, 10]) $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$, where $\mathcal{H}_+ = \mathrm{Dom}(\dot{A}^*)$ and

$$(f, g)_+ = (f, g) + (\dot{A}^* f, \dot{A}^* g), \quad f, g \in \mathrm{Dom}(A^*). \tag{2.1}$$

Let $\mathcal{R}$ be the *Riesz-Berezansky operator* $\mathcal{R}$ (see [2, 10]) which maps $\mathcal{H}_-$ onto $\mathcal{H}_+$ such that $(f, g) = (f, \mathcal{R}g)_+$ ($\forall f \in \mathcal{H}_+$, $g \in \mathcal{H}_-$) and $\|\mathcal{R}g\|_+ = \|g\|_-$. Note that identifying the space conjugate to $\mathcal{H}_\pm$ with $\mathcal{H}_\mp$, we get that if $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_-]$, then $\mathbb{A}^* \in [\mathcal{H}_+, \mathcal{H}_-]$. An operator $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ is called a *self-adjoint bi-extension* of a symmetric operator $\dot{A}$ if $\mathbb{A} = \mathbb{A}^*$ and $\mathbb{A} \supset \dot{A}$. Let $\mathbb{A}$ be a self-adjoint bi-extension of $\dot{A}$ and let the operator $\hat{A}$ in $\mathcal{H}$ be defined as follows:

$$\operatorname{Dom}(\hat{A}) = \{f \in \mathcal{H}_+ : \mathbb{A}f \in \mathcal{H}\}, \quad \hat{A} = \mathbb{A}\upharpoonright \operatorname{Dom}(\hat{A}).$$

The operator $\hat{A}$ is called the *quasi-kernel* of a self-adjoint bi-extension $\mathbb{A}$ (see [2, Section 2.1], [26]). According to the von Neumann Theorem (see [2, Theorem 1.3.1]) the domain of $\widehat{A}$, a self-adjoint extension of $\dot{A}$, can be expressed as

$$\operatorname{Dom}(\hat{A}) = \operatorname{Dom}(\dot{A}) \oplus (I + U)\mathfrak{N}_i, \tag{2.2}$$

where von Neumann's parameter U is both a $(\cdot)$-isometric as well as $(+)$-isometric operator from $\mathfrak{N}_i$ into $\mathfrak{N}_{-i}$ and

$$\mathfrak{N}_{\pm i} = \operatorname{Ker}(\dot{A}^* \mp iI)$$

are the deficiency subspaces of $\dot{A}$. A self-adjoint bi-extension $\mathbb{A}$ of a symmetric operator $\dot{A}$ is called *t-self-adjoint* (see [2, Definition 4.3.1]) if its quasi-kernel $\hat{A}$ is a self-adjoint operator in $\mathcal{H}$. An operator $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ is called the *quasi-self-adjoint bi-extension* of an operator T if $\mathbb{A} \supset T \supset \dot{A}$ and $\mathbb{A}^* \supset T^* \supset \dot{A}$.

We will be mostly interested in the following type of quasi-self-adjoint bi-extensions. Let T be a quasi-self-adjoint extension of $\dot{A}$ with nonempty resolvent set $\rho(T)$. A quasi-self-adjoint bi-extension $\mathbb{A}$ of an operator T is called (see [2, Definition 3.3.5]) a *$(*)$-extension* of T if $\operatorname{Re}\mathbb{A}$ is a t-self-adjoint bi-extension of $\dot{A}$. In what follows we assume that $\dot{A}$ has deficiency indices $(1, 1)$. In this case it is known [2] that every quasi-self-adjoint extension T of $\dot{A}$ admits $(*)$-extensions. The description of all $(*)$-extensions via the Riesz-Berezansky operator $\mathcal{R}$ can be found in [2, Section 4.3].

Recall that a linear operator T in a Hilbert space $\mathcal{H}$ is called **accretive** [15] if $\operatorname{Re}(Tf, f) \geq 0$ for all $f \in \operatorname{Dom}(T)$. We call an accretive operator T α-**sectorial** [15] if there exists a value of $\alpha \in (0, \pi/2)$ such that

$$(\cot\alpha)|\operatorname{Im}(Tf, f)| \leq \operatorname{Re}(Tf, f), \qquad f \in \operatorname{Dom}(T). \tag{2.3}$$

We say that the angle of sectoriality α is **exact** for an α-sectorial operator T if

$$\tan\alpha = \sup_{f \in \operatorname{Dom}(T)} \frac{|\operatorname{Im}(Tf, f)|}{\operatorname{Re}(Tf, f)}.$$

An accretive operator is called **extremal accretive** if it is not α-sectorial for any $\alpha \in (0, \pi/2)$. In what follows, when we say that an accretive operator is α-sectorial, we mean that α is its exact angle of sectoriality unless otherwise is specified.

The following definition is a "lite" version of the definition of L-system given for a scattering L-system with one-dimensional input–output space. It is tailored for the case when the symmetric operator of an L-system has deficiency indices $(1, 1)$. The general definition of an L-system can be found in [2, Definition 6.3.4].

Definition 1 An array

$$\Theta = \begin{pmatrix} \mathbb{A} & K & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix} \tag{2.4}$$

is called the **L-system** if:

(1) T is a dissipative ($Im(Tf, f) \geq 0$, $f \in \operatorname{Dom}(T)$) quasi-self-adjoint extension of a symmetric operator $\dot{A}$ with deficiency indices $(1, 1)$;
(2) $\mathbb{A}$ is a $(*)$-extension of T;
(3) $\operatorname{Im} \mathbb{A} = KK^*$, where $K \in [\mathbb{C}, \mathcal{H}_-]$ and $K^* \in [\mathcal{H}_+, \mathbb{C}]$.

Operators T and $\mathbb{A}$ are called the *main and state-space operators respectively* of the system Θ, and K is the *channel operator*. It is easy to see that the operator $\mathbb{A}$ of the system (2.4) can be chosen in such a way that $\operatorname{Im} \mathbb{A} = (\cdot, \chi)\chi$, $\chi \in \mathcal{H}_-$ and $Kc = c \cdot \chi$, $c \in \mathbb{C}$. A system Θ in (2.4) is called *minimal* if the operator $\dot{A}$ is a prime operator in $\mathcal{H}$, i.e., there exists no non-trivial reducing invariant subspace of $\mathcal{H}$ on which it induces a self-adjoint operator. Notice that minimal L-systems of the form (2.4) with one-dimensional input–output space were also considered in [6].

We associate with an L-system Θ the function

$$W_\Theta(z) = I - 2iK^*(\mathbb{A} - zI)^{-1}K, \quad z \in \rho(T), \tag{2.5}$$

which is called the **transfer function** of the L-system Θ. We also consider another function related to an L-system Θ called the **impedance function** and given by the formula

$$V_\Theta(z) = K^*(\operatorname{Re} \mathbb{A} - zI)^{-1}K. \tag{2.6}$$

The transfer function $W_\Theta(z)$ of the L-system Θ and function $V_\Theta(z)$ of the form (2.6) are connected by the following relations valid for $\operatorname{Im} z \neq 0$, $z \in \rho(T)$,

$$\begin{aligned} V_\Theta(z) &= i[W_\Theta(z) + I]^{-1}[W_\Theta(z) - I], \\ W_\Theta(z) &= (I + iV_\Theta(z))^{-1}(I - iV_\Theta(z)). \end{aligned}$$

The class of all Herglotz–Nevanlinna functions, that can be realized as impedance functions of L-systems, and connections with Weyl–Titchmarsh functions can be found in [2, 6, 11, 14] and references therein. In particular it is shown there that any impedance function $V_\Theta(z)$ admits the integral representation

$$V_\Theta(z) = Q + \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\sigma, \tag{2.7}$$

where Q is a real number and σ is an infinite Borel measure such that

$$\int_{\mathbb{R}} \frac{d\sigma(\lambda)}{1 + \lambda^2} < \infty.$$

3 Donoghue Classes and L-Systems with One-Dimensional Input–Output

Suppose that $\dot{A}$ is a closed prime densely defined symmetric operator with deficiency indices $(1, 1)$. Assume also that $T \neq T^*$ is a maximal dissipative extension of $\dot{A}$,

$$\operatorname{Im}(Tf, f) \geq 0, \quad f \in \operatorname{Dom}(T).$$

Since $\dot{A}$ is symmetric, its dissipative extension T is automatically quasi-self-adjoint [2], that is,

$$\dot{A} \subset T \subset \dot{A}^*,$$

and hence, (see [6])

$$g_+ - \kappa g_- \in \operatorname{Dom}(T) \quad \text{for some } |\kappa| < 1, \tag{3.1}$$

where $g_\pm \in \mathfrak{N}_{\pm i} = \operatorname{Ker}(\dot{A}^* \mp iI)$ and $\|g_\pm\| = 1$. Throughout this paper κ will be referred to as the **von Neumann parameter** of the operator T. The next lemma contains a characterization of sectorial operators in terms of the modulus of von Neumann's parameter.

Lemma 2 *If T is an α-sectorial operator with $\alpha \in (0, \pi/2)$, then its von Neumann's parameter κ cannot equal zero.*

Proof Assume the contrary, let T be an α-sectorial operator in a Hilbert space $\mathcal{H}$ with $\alpha \in (0, \pi/2)$ and the von Neumann parameter $\kappa = 0$. Then (3.1) implies (see [2]) that there exists a non-zero vector $x \in \operatorname{Dom}(T)$ such that $Tx = ix$, $x \neq 0$. Moreover,

$$(Tx, x) = (ix, x) = i\|x\|^2 \quad \text{and} \quad (x, Tx) = (x, ix) = -i\|x\|^2.$$

Then

$$\operatorname{Re}(Tx, x) = \frac{(Tx, x) + (x, Tx)}{2} = 0$$

and

$$\operatorname{Im}(Tx, x) = \frac{(Tx, x) - (x, Tx)}{2i} = \frac{i\|x\|^2 + i\|x\|^2}{2i} = \|x\|^2.$$

But T is α-sectorial and hence (2.3) takes place implying

$$0 \leq (\cot\alpha)|\operatorname{Im}(Tx, x)| \leq \operatorname{Re}(Tx, x) = 0.$$

This yields $0 \leq (\cot\alpha)\|x\|^2 \leq 0$ or $(\cot\alpha)\|x\|^2 = 0$. Since neither $\cot\alpha$ nor $\|x\|$ can equal zero, we reached a contradiction. Therefore, $\kappa \neq 0$. □

Recall that Donoghue [12] introduced a concept of the Herglotz–Nevanlinna function $M_{(\dot{A},A)}(z)$ associated with the pair $(\dot{A}, A)$ by

$$\begin{aligned} &M_{(\dot{A},A)}(z) = ((Az+I)(A-zI)^{-1}g_+, g_+), \quad z \in \mathbb{C}_+, \\ &g_+ \in \operatorname{Ker}(\dot{A}^* - iI), \quad \|g_+\| = 1, \end{aligned}$$

where $\dot{A}$ is a symmetric operator with deficiency indices $(1, 1)$, and A is its self-adjoint extension. Let $\mathfrak{N}$ (see [9]) be a class of all Herglotz–Nevanlinna functions $M(z)$ that admit the representation

$$M(z) = \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right) d\sigma, \tag{3.2}$$

where σ is an infinite Borel measure with

$$\int_{\mathbb{R}} \frac{d\sigma(\lambda)}{1+\lambda^2} < \infty.$$

Following our earlier developments in [6, 9, 18, 19] denote by $\mathfrak{M}$ the **Donoghue class** of all analytic mappings $M(z)$ from $\mathbb{C}_+$ into itself that admits the representation (3.2) and has a property

$$\int_{\mathbb{R}} \frac{d\sigma(\lambda)}{1+\lambda^2} = 1, \quad \text{equivalently,} \quad M(i) = i.$$

It is known [12–14, 17] that $M(z) \in \mathfrak{M}$ if and only if $M(z)$ can be realized as the Weyl–Titchmarsh function $M_{(\dot{A},A)}(z)$ associated with the pair $(\dot{A}, A)$. Furthermore, we say (see [6]) that a function $M(z) \in \mathfrak{N}$ belongs to the **generalized Donoghue class** $\mathfrak{M}_\kappa$, $(0 \leq \kappa < 1)$ if in the representation (3.2)

$$\int_{\mathbb{R}} \frac{d\sigma(\lambda)}{1+\lambda^2} = \frac{1-\kappa}{1+\kappa}, \quad \text{equivalently,} \quad M(i) = i\,\frac{1-\kappa}{1+\kappa}. \tag{3.3}$$

Similarly (see [7]), a function $M(z) \in \mathfrak{N}$ belongs to the **generalized Donoghue class** $\mathfrak{M}_\kappa^{-1}$ if in the representation (3.2)

$$\int_{\mathbb{R}} \frac{d\sigma(\lambda)}{1+\lambda^2} = \frac{1+\kappa}{1-\kappa}, \quad \text{equivalently,} \quad M(i) = i\,\frac{1+\kappa}{1-\kappa}. \tag{3.4}$$

Clearly, when $\kappa = 0$ the generalized Donoghue classes $\mathfrak{M}_\kappa$ and $\mathfrak{M}_\kappa^{-1}$ coincide with the Donoghue class $\mathfrak{M}$, that is $\mathfrak{M}_0 = \mathfrak{M}_0^{-1} = \mathfrak{M}$. If $M(z)$ is an arbitrary function from $\mathfrak{N}$ with a normalization condition

$$\int_{\mathbb{R}} \frac{d\sigma(\lambda)}{1+\lambda^2} = a, \tag{3.5}$$

for some $a > 0$, then it is easy to see that $M(z) \in \mathfrak{M}$ if and only if $a = 1$. Also, if $a < 1$, then $M(z) \in \mathfrak{M}_\kappa$ with

$$\kappa = \frac{1-a}{1+a}, \tag{3.6}$$

and if $a > 1$, then $M(z) \in \mathfrak{M}_\kappa^{-1}$ with

$$\kappa = \frac{a-1}{1+a}. \tag{3.7}$$

Hypothesis 3 *Suppose that $T \neq T^*$ is a maximal dissipative extension of a symmetric operator $\dot{A}$ with deficiency indices* $(1, 1)$. *Assume, in addition, that A is a self-adjoint extension of $\dot{A}$. Suppose, that the deficiency elements $g_\pm \in Ker\,(\dot{A}^* \mp iI)$ are normalized, $\|g_\pm\| = 1$, and chosen in such a way that*

$$g_+ - g_- \in \operatorname{Dom}(A) \;\; \textit{and} \;\; g_+ - \kappa g_- \in \operatorname{Dom}(T) \;\; \textit{for some} \quad |\kappa| < 1. \tag{3.8}$$

It is known [17] that if $\kappa = 0$, then quasi-self-adjoint extension T coincides with the restriction of the adjoint operator $\dot{A}^*$ on

$$\operatorname{Dom}(T) = \operatorname{Dom}(\dot{A}) \dot{+} \operatorname{Ker}\,(\dot{A}^* - iI).$$

Similar to Hypothesis 3 it is convenient to adopt the “anti-Hypothesis".

Hypothesis 4 *Suppose that $T \neq T^*$ is a maximal dissipative extension of a symmetric operator $\dot{A}$ with deficiency indices* $(1, 1)$. *Assume, in addition, that A is a self-adjoint extension of $\dot{A}$. Suppose, that the deficiency elements $g_\pm \in Ker\,(\dot{A}^* \mp iI)$ are normalized, $\|g_\pm\| = 1$, and chosen in such a way that*

$$g_+ + g_- \in \operatorname{Dom}(A) \;\; \textit{and} \;\; g_+ - \kappa g_- \in \operatorname{Dom}(T) \;\; \textit{for some} \quad |\kappa| < 1. \tag{3.9}$$

Remark 5 Without loss of generality, in what follows we assume that κ is real and $0 \le \kappa < 1$: if $\kappa = |\kappa| e^{i\theta}$, change (the basis) g_- to $e^{i\theta} g_-$ in the deficiency subspace $\operatorname{Ker}\,(\dot{A}^* + iI)$.

This remark means the following: let

$$\Theta = \begin{pmatrix} & \mathbb{A} & & K & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & & & \mathbb{C} \end{pmatrix} \tag{3.10}$$

be a minimal L-system with one-dimensional input–output space $\mathbb{C}$. If the main operator T of Θ is parameterized with a *complex* von Neumann's parameter κ that corresponds to a chosen normalized pair of deficiency vectors g_+ and g_-, then we can change the deficiency basis as explained in Remark 5 and represent T using real value $|\kappa|$ with respect to the new deficiency basis. This procedure will change the parameter U of the quasi-kernel $\hat{A}$ of $\operatorname{Re} \mathbb{A}$ in (2.2) and ultimately the way $\mathbb{A}$ is described.

Thus, for the remainder of this paper (unless otherwise is specified) we will consider L-systems (3.10) such that κ is real and $0 \le \kappa < 1$.

Definition 6 We say that an L-system Θ of the form (3.10) *satisfies Hypothesis* 3 (or 4) if its main operator T and the quasi-kernel $\hat{A}$ of Re $\mathbb{A}$ satisfy the conditions of Hypothesis 3 (or 4) for a fixed set of deficiency vectors of the symmetric operator $\dot{A}$.

Let Θ be a minimal L-system of the form (3.10) that satisfies Hypothesis 3. It is shown in [6] that the impedance function $V_\Theta(z)$ can be represented as

$$V_\Theta(z) = \left(\frac{1-\kappa}{1+\kappa}\right) V_{\Theta_0}(z), \tag{3.11}$$

where $V_{\Theta_0}(z)$ is the impedance function of an L-system Θ_0 with the same set of conditions but with $\kappa_0 = 0$, where κ_0 is the von Neumann parameter of the main operator T_0 of Θ_0.

Let Θ_1 and Θ_2 be two minimal L-system of the form (3.10) whose components satisfy Hypothesis 3 and Hypothesis 4, respectively. Then it was proved in [7, Lemma 5.1] that the impedance functions $V_{\Theta_1}(z)$ and $V_{\Theta_2}(z)$ admit the integral representation

$$V_{\Theta_k}(z) = \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\sigma_k(t), \quad k = 1, 2. \tag{3.12}$$

Now let us consider a minimal L-system Θ of the form (3.10) that satisfies Hypothesis 3. Let also

$$\Theta_\alpha = \begin{pmatrix} \mathbb{A}_\alpha & K_\alpha & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix}, \quad \alpha \in [0, \pi), \tag{3.13}$$

be a one parametric family of L-systems such that

$$W_{\Theta_\alpha}(z) = W_\Theta(z) \cdot (-e^{2i\alpha}), \quad \alpha \in [0, \pi). \tag{3.14}$$

The existence and structure of Θ_α were described in details in [2, Section 8.3]. In particular, it was shown that the L-system Θ and Θ_α share the same main operator T and that

$$V_{\Theta_\alpha}(z) = \frac{\cos\alpha + (\sin\alpha) V_\Theta(z)}{\sin\alpha - (\cos\alpha) V_\Theta(z)}. \tag{3.15}$$

Let Θ be a minimal L-system Θ of the form (3.10) that satisfies Hypothesis 3. Also let Θ_α be a one parametric family of L-systems given by (3.13), (3.14). It was shown in [7, Theorem 5.2] that in this case the impedance function $V_{\Theta_\alpha}(z)$ has an integral representation

$$V_{\Theta_\alpha}(z) = \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\sigma_\alpha(t)$$

if and only if $\alpha = 0$ or $\alpha = \pi/2$.

The next result describes the relationship between two L-systems of the form (3.10) that comply with different hypotheses.

Let

$$\Theta_1 = \begin{pmatrix} \mathbb{A}_1 & K_1 & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix} \tag{3.16}$$

be a minimal L-system whose main operator T and the quasi-kernel $\hat{A}_1$ of $\operatorname{Re}\mathbb{A}_1$ satisfy the conditions of Hypothesis 3 and let

$$\Theta_2 = \begin{pmatrix} \mathbb{A}_2 & K_2 & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix} \tag{3.17}$$

be another minimal L-system with the same operators $\dot{A}$ and T as Θ_1 but with the quasi-kernel $\hat{A}_2$ of $\operatorname{Re}\mathbb{A}_2$ that satisfies Hypothesis 4. It was shown in [7, Theorem 5.3] that

$$W_{\Theta_1}(z) = -W_{\Theta_2}(z), \quad z \in \mathbb{C}_+ \cap \rho(T) \tag{3.18}$$

and

$$V_{\Theta_1}(z) = -\frac{1}{V_{\Theta_2}(z)}, \quad z \in \mathbb{C}_+ \cap \rho(T). \tag{3.19}$$

4 L-Systems with Schrödinger Operator T_h and Their Impedance Functions

Let $\mathcal{H} = L_2(\ell, +\infty)$ and $l(y) = -y'' + q(x)y$, where q is a real locally summable function. Suppose that the minimal symmetric operator

$$\begin{cases} \dot{A}y = -y'' + q(x)y \\ y(\ell) = y'(\ell) = 0 \end{cases} \tag{4.1}$$

has deficiency indices (1,1). Let D^* be the set of functions locally absolutely continuous together with their first derivatives such that $l(y) \in L_2(\ell, +\infty)$. Consider $\mathcal{H}_+ = \operatorname{Dom}(\dot{A}^*) = D^*$ with the scalar product

$$(y, z)_+ = \int_a^\infty \left(y(x)\overline{z(x)} + l(y)\overline{l(z)} \right) dx, \quad y,\ z \in D^*.$$

Let $\mathcal{H}_+ \subset L_2(\ell, +\infty) \subset \mathcal{H}_-$ be the corresponding triplet of Hilbert spaces. Consider the operators (cf. [21])

$$\begin{cases} T_h y = l(y) = -y'' + q(x)y \\ hy(\ell) - y'(\ell) = 0 \end{cases}, \quad \begin{cases} T_h^* y = l(y) = -y'' + q(x)y \\ \overline{h}y(\ell) - y'(\ell) = 0 \end{cases}. \tag{4.2}$$

Let $\dot{A}$ be a symmetric operator of the form (4.1) with deficiency indices (1,1), generated by the differential expression $l(y) = -y'' + q(x)y$. Let also $\varphi_k(x, \lambda)(k = 1, 2)$ be the solutions of the following Cauchy problems:

$$\begin{cases} l(\varphi_1) = \lambda\varphi_1 \\ \varphi_1(\ell, \lambda) = 0 \\ \varphi_1'(\ell, \lambda) = 1 \end{cases}, \qquad \begin{cases} l(\varphi_2) = \lambda\varphi_2 \\ \varphi_2(\ell, \lambda) = -1 \\ \varphi_2'(\ell, \lambda) = 0 \end{cases}.$$

It is well known [20] that there exists a Weyl function $m_\infty(\lambda)$ such that

$$\varphi(x, \lambda) = \varphi_2(x, \lambda) + m_\infty(\lambda)\varphi_1(x, \lambda)$$

belongs to $L_2(\ell, +\infty)$.

Now we shall construct an L-system associated with a non-self-adjoint Schrödinger operator T_h. It was shown in [2, 4] that the set of all $(*)$-extensions of the non-self-adjoint Schrödinger operator T_h of the form (4.2) in $L_2(\ell, +\infty)$ can be represented as

$$\begin{aligned} \mathbb{A}_{\mu,h}\, y &= -y'' + q(x)y - \frac{1}{\mu - h}\,[y'(\ell) - hy(\ell)]\,[\mu\delta(x - \ell) + \delta'(x - \ell)], \\ \mathbb{A}^*_{\mu,h}\, y &= -y'' + q(x)y - \frac{1}{\mu - \overline{h}}\,[y'(\ell) - \overline{h}y(\ell)]\,[\mu\delta(x - \ell) + \delta'(x - \ell)]. \end{aligned} \tag{4.3}$$

Moreover, the formulas (4.3) establish a one-to-one correspondence between the set of all $(*)$-extensions of the Schrödinger operator T_h of the form (4.2) and all values $\mu \in \mathbb{R} \cup \{\infty\}$. One can easily check that the $(*)$-extension $\mathbb{A}_{\mu,h}$ in (4.3) of the non-self-adjoint dissipative Schrödinger operator T_h, $(\operatorname{Im} h > 0)$ of the form (4.2) satisfies the condition

$$\operatorname{Im} \mathbb{A}_{\mu,h} = \frac{\mathbb{A}_{\mu,h} - \mathbb{A}^*_{\mu,h}}{2i} = (\cdot, g)g,$$

where

$$g = \frac{(\operatorname{Im} h)^{\frac{1}{2}}}{|\mu - h|}\,[\mu\delta(x - \ell) + \delta'(x - \ell)], \tag{4.4}$$

$\delta(x-\ell)$ and $\delta'(x-\ell)$ are the delta-function and its derivative at the point ℓ, respectively. Furthermore,

$$(y, g) = \frac{(\operatorname{Im} h)^{\frac{1}{2}}}{|\mu - h|}\,[\mu y(\ell) - y'(\ell)],$$

$y \in \mathcal{H}_+$, $g \in \mathcal{H}_-$, where $\mathcal{H}_+ \subset L_2(\ell, +\infty) \subset \mathcal{H}_-$ is the triplet of Hilbert spaces introduced above.

Let $y\in\operatorname{Dom}(T_h)$, then $y'(\ell)=hy(\ell)$ and

$$\begin{aligned}\operatorname{Im}\mathbb{A}_{\mu,h}\,y=\operatorname{Im}T_h y&=(y,g)g=\frac{(\operatorname{Im}h)^{\frac12}}{|\mu-h|}\,[\mu y(\ell)-y'(\ell)]g\\&=\frac{(\operatorname{Im}h)^{\frac12}}{|\mu-h|}\,[\mu y(\ell)-hy(\ell)]g\\&=\frac{(\operatorname{Im}h)^{\frac12}}{|\mu-h|}\,(\mu-h)y(\ell)\cdot\frac{(\operatorname{Im}h)^{\frac12}}{|\mu-h|}\,[\mu\delta(x-\ell)+\delta'(x-\ell)]\\&=y(\ell)\frac{(\operatorname{Im}h)(\mu-h)}{|\mu-h|^2}\,[\mu\delta(x-\ell)+\delta'(x-\ell)].\end{aligned}$$

Consequently,

$$\operatorname{Im}(T_h y,y)=y(\ell)\frac{(\operatorname{Im}h)(\mu-h)}{|\mu-h|^2}\cdot(\mu-\bar h)\overline{y(\ell)}=(\operatorname{Im}h)|y(\ell)|^2. \tag{4.5}$$

Having in mind (4.5) we will call $\operatorname{Im}h$ the **coefficient of dissipation** and denote it by $\mathcal{D}=\operatorname{Im}h$.

It was also shown in [2] that the quasi-kernel $\hat A_\xi$ of $\operatorname{Re}\mathbb{A}_{\mu,h}$ is given by

$$\begin{cases}\hat A_\xi y=-y''+q(x)y\\ y'(\ell)=\xi y(\ell)\end{cases},\quad\text{where}\quad \xi=\frac{\mu\operatorname{Re}h-|h|^2}{\mu-\operatorname{Re}h}. \tag{4.6}$$

Let $E=\mathbb{C}$, $Kc=cg$ $(c\in\mathbb{C})$. It is clear that

$$K^*y=(y,g),\quad y\in\mathcal{H}_+, \tag{4.7}$$

and $\operatorname{Im}\mathbb{A}_{\mu,h}=KK^*$. Therefore, the array

$$\Theta_{\mu,h}=\begin{pmatrix}\mathbb{A}_{\mu,h} & K & 1\\ \mathcal{H}_+\subset L_2(\ell,+\infty)\subset\mathcal{H}_- & & \mathbb{C}\end{pmatrix}, \tag{4.8}$$

is an L-system with the main operator $\mathbb{A}_{\mu,h}$ of the form (4.3) with the channel operator K given by (4.7). We will say that an L-system $\Theta_{\mu,h}$ of the form (4.8) has the coefficient of dissipation $\mathcal{D}$ if its main operator T_h has the coefficient of dissipation $\mathcal{D}=\operatorname{Im}h$. It was shown in [2, 4] that the transfer and impedance functions of $\Theta_{\mu,h}$ can be evaluated as

$$W_{\Theta_{\mu,h}}(z)=\frac{\mu-h}{\mu-\overline{h}}\,\frac{m_\infty(z)+\overline{h}}{m_\infty(z)+h}, \tag{4.9}$$

and

$$V_{\Theta_{\mu,h}}(z)=\frac{(m_\infty(z)+\mu)\operatorname{Im}h}{(\mu-\operatorname{Re}h)\,m_\infty(z)+\mu\operatorname{Re}h-|h|^2}. \tag{4.10}$$

Suppose that the main operator T_h of the L-system (4.8) has the von Neumann representation (3.1) with the parameter κ related to some normalized deficiency basis $g_\pm$. It was shown in [6] that if the point $z = i$ belongs to the resolvent set $\rho(T_h)$, then in this case (by (4.9))

$$|\kappa| = \frac{1}{|W_{\Theta_{\mu,h}}(i)|} = \left|\frac{\mu - \overline{h}}{\mu - h} \cdot \frac{m_\infty(i) + h}{m_\infty(i) + \overline{h}}\right| = \left|\frac{m_\infty(i) + h}{m_\infty(i) + \overline{h}}\right|. \tag{4.11}$$

Taking into account that $\operatorname{Im} h > 0$ and $\operatorname{Im} m_\infty(i) < 0$ one can easily see that $|\kappa| < 1$ in (4.11).

Remark 7 Now suppose $h \neq -m_\infty(i)$ and hence $\kappa \neq 0$. If $\Theta_{\mu,h}$ satisfies Hypothesis 3, then

$$\kappa = \frac{1}{W_{\Theta_{\mu,h}}(i)} = \frac{\mu - \overline{h}}{\mu - h} \cdot \frac{m_\infty(i) + h}{m_\infty(i) + \overline{h}} \tag{4.12}$$

is real.

We can reverse the logic and try to find a value of parameter μ for which κ in (4.12) is real and thus $\kappa = |\kappa|$. This can be achieved if we set

$$\frac{\mu - \overline{h}}{\mu - h} = e^{-i\alpha}, \tag{4.13}$$

where $\alpha \in (-\pi, \pi]$ is such that

$$\frac{m_\infty(i) + h}{m_\infty(i) + \overline{h}} = \left|\frac{m_\infty(i) + h}{m_\infty(i) + \overline{h}}\right| e^{i\alpha}.$$

Clearly, such a choice of the angle α will make κ real and $0 < \kappa < 1$. That is,

$$\kappa = \left|\frac{m_\infty(i) + h}{m_\infty(i) + \overline{h}}\right|. \tag{4.14}$$

Solving (4.14) for μ yields

$$\mu = \mu_1 = \frac{e^{i\alpha}\overline{h} - h}{e^{i\alpha} - 1}, \quad \text{where} \quad \alpha = \operatorname{Arg}\left(\frac{m_\infty(i) + h}{m_\infty(i) + \overline{h}}\right). \tag{4.15}$$

Therefore, we can describe an L-system

$$\Theta_{\mu_1,h} = \begin{pmatrix} \mathbb{A}_{\mu_1,h} & K_1 & 1 \\ \mathcal{H}_+ \subset L_2(\ell, +\infty) \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix}, \tag{4.16}$$

with main Schrödinger operator T_h that satisfies Hypothesis 3 with $0 < \kappa < 1$. The state-space operator $\mathbb{A}_{\mu_1,h}$ of this L-system will be uniquely defined via (4.3) by the parameter h of T_h and parameter μ_1 given by (4.15).

Similarly we can give the description of an L-system

$$\Theta_{\mu_2,h} = \begin{pmatrix} & \mathbb{A}_{\mu_2,h} & K_2 \ 1 \\ \mathcal{H}_+ \subset L_2(\ell,+\infty) \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix}, \tag{4.17}$$

with the same main operator T_h that satisfies the conditions of Hypothesis 4 with $0 < \kappa < 1$. Applying (3.18) yields

$$\mu = \mu_2 = \frac{e^{i\alpha}\bar{h} + h}{e^{i\alpha} + 1}, \quad \text{where} \quad \alpha = \operatorname{Arg}\left(\frac{m_\infty(i) + h}{m_\infty(i) + \bar{h}}\right). \tag{4.18}$$

The state-space operator $\mathbb{A}_{\mu_2,h}$ from Hypothesis 4 satisfying L-system will be uniquely defined via (4.3) by the parameter h of T_h and parameter μ_2 given by (4.18). As shown in [7],

$$V_{\Theta_{\mu_1,h}}(z) \in \mathfrak{M}_\kappa \quad \textit{and} \quad V_{\Theta_{\mu_2,h}}(z) \in \mathfrak{M}_\kappa^{-1}.$$

We will rely on formula (4.11) to prove the following result.

Theorem 8 *Let $\Theta_{\mu,h}$ be a Schrödinger L-system (4.8) with the main operator T_h of the form (4.2). Then for any $\mu \in \mathbb{R} \cup \{\infty\}$ the impedance function $V_{\Theta_{\mu,h}}(z)$ belongs to the class $\mathfrak{M}$ if and only if $h = -m_\infty(i)$.*

Proof It is well known (see [6, 7]) that the impedance function $V_{\Theta_{\mu,h}}(z)$ belongs to the class $\mathfrak{M}$ if and only if it has the integral representation

$$V_{\Theta_{\mu,h}}(z) = \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right) d\sigma(\lambda), \quad \int_{\mathbb{R}} \frac{d\sigma(\lambda)}{1+\lambda^2} = 1. \tag{4.19}$$

Suppose that the Schrödinger L-system $\Theta_{\mu,h}$ contains the main operator T_h such that $h = -m_\infty(i)$ and arbitrary $\mu \in \mathbb{R} \cup \{\infty\}$. Then (4.10) yields

$$\begin{aligned} V_{\Theta_{\mu,h}}(i) &= \frac{-(m_\infty(z) + \mu)\operatorname{Im} m_\infty(i)}{(\mu + \operatorname{Re} m_\infty(i))\, m_\infty(i) - \mu \operatorname{Re} m_\infty(i) - |m_\infty(i)|^2} \\ &= \frac{-(m_\infty(z) + \mu)\operatorname{Im} m_\infty(i)}{\mu\,(\operatorname{Re} m_\infty(i) + i \operatorname{Im} m_\infty(i)) + \operatorname{Re} m_\infty(i) \cdot m_\infty(i) - \mu \operatorname{Re} m_\infty(i) - |m_\infty(i)|^2} \\ &= \frac{-(m_\infty(z) + \mu)\operatorname{Im} m_\infty(i)}{i\mu \operatorname{Im} m_\infty(i) + \operatorname{Re} m_\infty(i) \cdot m_\infty(i) - |m_\infty(i)|^2} \\ &= \frac{-(m_\infty(z) + \mu)\operatorname{Im} m_\infty(i)}{i\mu \operatorname{Im} m_\infty(i) + \left(\operatorname{Re} m_\infty(i) - \overline{m_\infty(i)}\right) m_\infty(i)} \\ &= \frac{-(m_\infty(z) + \mu)\operatorname{Im} m_\infty(i)}{i\mu \operatorname{Im} m_\infty(i) + (\operatorname{Re} m_\infty(i) - \operatorname{Re} m_\infty(i) + i \operatorname{Im} m_\infty(i))\, m_\infty(i)} \\ &= \frac{-(m_\infty(z) + \mu)\operatorname{Im} m_\infty(i)}{i\mu \operatorname{Im} m_\infty(i)\,(\mu + m_\infty(i))} = -\frac{1}{i} = i. \end{aligned}$$

Therefore, $V_{\Theta_{\mu,h}}(z)$ admits the integral representation (4.19) and hence belongs to the class $\mathfrak{M}$.

Conversely, let $V_{\Theta_{\mu,h}}(z)$ belong to the class $\mathfrak{M}$ and hence has the integral representation (4.19). Then $V_{\Theta_{\mu,h}}(z) \in \mathfrak{M}$ and consequently (see [6]) $\kappa = 0$, where κ is the von Neumann parameter of T_h. In this case (4.11) implies that $h = -m_\infty(i)$. □

Clearly, it follows from (4.11) that $\kappa = 0$ if and only if $h = -m_\infty(i)$. Therefore,

$$\begin{cases} T_h y = -y'' + q(x)y \\ y'(\ell) = hy(\ell) \end{cases} \quad \text{has } \kappa = 0 \text{ if and only if } \; h = -m_\infty(i). \tag{4.20}$$

The state-space operator $\mathbb{A}_{\mu,h}$ of the L-system (4.8) (and its adjoint $\mathbb{A}^*_{\mu,h}$) in this case will depend on the parameter μ only and take the form

$$\begin{aligned} \mathbb{A}_{\mu,-m_\infty(i)}\, y &= -y'' + q(x)y - \frac{y'(\ell) + m_\infty(i)y(\ell)}{\mu + m_\infty(i)}[\mu\delta(x-\ell) + \delta'(x-\ell)], \\ \mathbb{A}^*_{\mu,-m_\infty(i)}\, y &= -y'' + q(x)y - \frac{y'(\ell) + m_\infty(-i)y(\ell)}{\mu + m_\infty(-i)}[\mu\delta(x-\ell) + \delta'(x-\ell)]. \end{aligned} \tag{4.21}$$

It can be checked directly that in this case the impedance function $V_{\Theta_{\mu,h}}(\lambda)$ belongs to the class $\mathfrak{M}$ (see also [6]).

The following theorem is similar to Theorem 8 result for the class $\mathfrak{M}_\kappa$.

Theorem 9 *Let $0 < a < 1$ and $\kappa = \frac{1-a}{1+a}$. Also let $\Theta_{\mu,h}$ be a Schrödinger L-system (4.8) with the main operator T_h of the form (4.2) that has the modulus of von Neumann's parameter κ. Then the impedance function $V_{\Theta_{\mu,h}}(z)$ belongs to the class $\mathfrak{M}_\kappa$ if and only if either*

$$h = -Re\, m_\infty(i) - \left(\frac{i}{a}\right) \operatorname{Im} m_\infty(i), \tag{4.22}$$

or

$$h = \frac{a^2d^2\mu - d^2(c+\mu) - c(c+\mu)^2}{a^2d^2 + (c+\mu)^2} - i\,\frac{ad^3 + ad(c+\mu)^2}{a^2d^2 + (c+\mu)^2}, \tag{4.23}$$

where $c = Re\, m_\infty(i)$ and $d = \operatorname{Im} m_\infty(i)$.

Proof It is well known (see [7]) that the impedance function $V_{\Theta_{\mu,h}}(z)$ belongs to the class $\mathfrak{M}_\kappa$ for $\kappa = \frac{1-a}{1+a}$, $0 < a < 1$ if and only if it has the integral representation

$$V_{\Theta_{\mu,h}}(z) = \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right) d\sigma(\lambda), \quad \int_{\mathbb{R}} \frac{d\sigma(\lambda)}{1+\lambda^2} = a, \quad 0 < a < 1, \tag{4.24}$$

Suppose that the impedance function $V_{\Theta_{\mu,h}}(z) \in \mathfrak{M}_\kappa$ and thus has the integral representation (4.24) with $0 < a < 1$. Then, (4.10) and (4.24) imply

$$V_{\Theta_{\mu,h}}(i) = \frac{(m_\infty(i) + \mu)\operatorname{Im} h}{(\mu - \operatorname{Re} h)\, m_\infty(i) + \mu \operatorname{Re} h - |h|^2} = ai. \tag{4.25}$$

Setting

$$c = \operatorname{Re} m_\infty(i), \quad d = \operatorname{Im} m_\infty(i), \quad x = \operatorname{Re} h, \quad y = \operatorname{Im} h, \tag{4.26}$$

transforms (4.25) into

$$\frac{cy + \mu y + dyi}{(\mu - x)c + \mu x - x^2 - y^2 + (\mu d - xd)i} = ai. \tag{4.27}$$

Cross multiplying (4.27) and equating the real parts yields $cy + \mu y = axd - a\mu d$, or

$$y = \frac{ad(x - \mu)}{c + \mu}. \tag{4.28}$$

Equating the imaginary parts in (4.27) leads to

$$dy = ac(\mu - x) + a\mu x - ax^2 - ay^2.$$

Substituting (4.28) into the above equation results in

$$\frac{d^2(\mu - x)}{c + \mu} + c(\mu - x) + x(\mu - x) - \frac{a^2 d^2 (\mu - x)^2}{(c + \mu)^2} = 0,$$

or

$$(\mu - x)\left(\frac{d^2}{c + \mu} + c + x - \frac{a^2 d^2 (\mu - x)}{(c + \mu)^2}\right) = 0.$$

Exploring the first solution of the above equation when $\mu = x$ and applying (4.28) leads us to $x = -c$ and $y = -d/a$, or

$$\operatorname{Re} h = -\operatorname{Re} m_\infty(i) \quad \text{and} \quad \operatorname{Im} h = -\left(\frac{i}{a}\right) \operatorname{Im} m_\infty(i),$$

that confirms (4.22).

Assuming that $\mu \neq x$ and setting the second factor to be zero yields

$$a^2 d^2 (\mu - x) - c(c + \mu)^2 - x(c + \mu)^2 - d^2(c + \mu) = 0,$$

that we solve for x to obtain

$$x = \frac{a^2 d^2 \mu - d^2(c + \mu) - c(c + \mu)^2}{a^2 d^2 + (c + \mu)^2}. \tag{4.29}$$

Substituting (4.29) into (4.28) results in

$$y = -\frac{ad^3 + ad(c+\mu)^2}{a^2d^2 + (c+\mu)^2}. \tag{4.30}$$

Keeping in mind that $x = \operatorname{Re} h$ and $y = \operatorname{Im} h$ we confirm (4.23).

Conversely, suppose either (4.22) or (4.23) hold. If (4.22) is true, then substituting this value of h and $\mu = -\operatorname{Re} m_\infty(i)$ into the left side of (4.25) we obtain that $V_{\Theta_{\mu,h}}(i) = ai$. In case if (4.23) is true we also substitute the value of h in (4.25) and get that $V_{\Theta_{\mu,h}}(i) = ai$ for any real $\mu \neq -\operatorname{Re} m_\infty(i)$. Combining this normalization condition of $V_{\Theta_{\mu,h}}(z)$ with its known integral representation we conclude that $V_{\Theta_{\mu,h}}(z) \in \mathfrak{M}_\kappa$ and is described by (4.24). □

A very similar result takes place for the class $\mathfrak{M}_\kappa^{-1}$.

Theorem 10 *Let $a > 1$ and $\kappa = \frac{a-1}{1+a}$. Also, let $\Theta_{\mu,h}$ be a Schrödinger L-system (4.8) with the main operator T_h of the form (4.2) that has the modulus of von Neumann's parameter equal κ. Then the impedance function $V_{\Theta_{\mu,h}}(z)$ belongs to the class $\mathfrak{M}_\kappa^{-1}$ with $\kappa = \frac{a-1}{1+a}$ if and only if either (4.22) or (4.23) hold true for $a > 1$.*

Proof It is well known (see [7]) that the impedance function $V_{\Theta_{\mu,h}}(z)$ belongs to the class $\mathfrak{M}_\kappa^{-1}$ for $\kappa = \frac{a-1}{1+a}$, $a > 1$ if and only if it has the integral representation

$$V_{\Theta_{\mu,h}}(z) = \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right) d\sigma(\lambda), \quad \int_{\mathbb{R}} \frac{d\sigma(\lambda)}{1+\lambda^2} = a, \quad 1 < a. \tag{4.31}$$

The rest of the proof of Theorem 10 resembles the one of Theorem 9 taking into account that $a > 1$ this time. Suppose that the impedance function $V_{\Theta_{\mu,h}}(z) \in \mathfrak{M}_\kappa^{-1}$ with κ given by (3.7) and hence has the integral representation (4.31) with $a > 1$. Then, as we have shown in the proof of Theorem 9, relations (4.10) and (4.24) imply (4.25) for $a > 1$. Following the steps in the proof of Theorem 9 we use (4.25) to obtain (4.29) and (4.30) for $a > 1$ thus confirming either (4.22) or (4.23).

Conversely, suppose either (4.22) or (4.23) hold true for $a > 1$. Following the steps of the proof of the necessity in Theorem 9 we obtain that $V_{\Theta_{\mu,h}}(i) = ai$ for $a > 1$ and conclude that $V_{\Theta_{\mu,h}}(z) \in \mathfrak{M}_\kappa^{-1}$. □

5 On von Neumann's Parameter of Extremal Schrödinger Operator T_h

Recall that, the Phillips-Kato extension problem is about giving a description of all maximal accretive and/or sectorial extensions A of a non-negative symmetric operator $\dot{A}$ such that $\dot{A} \subset A \subset \dot{A}^*$. In this and the next sections we discuss the Phillips-Kato extension problem for a non-negative Schrödinger operator with deficiency indices (1, 1) in $\mathcal{H} = L^2(\ell, \infty)$. For the one-dimensional Schrödinger operator T_h ($\operatorname{Im} h > 0$) on the semi-axis the Phillips-Kato extension problem in restricted sense was solved by one of the authors in [23] (see also [1, 3, 22]). The solution in terms of boundary values is presented in Theorem 11 below. Our goal, however, is in presenting the necessary condition to the existence of solution for the Phillips-Kato problem in terms of the modulus of von Neumann's parameter.

Suppose that the symmetric operator $\dot{A}$ of the form (4.1) with deficiency indices (1,1) is nonnegative, i.e., $(\dot{A}f, f) \geq 0$ for all $f \in \mathrm{Dom}(\dot{A})$.

Theorem 11 ([22, 23], see also [3]) *Let $\dot{A}$ be a nonnegative symmetric Schrödinger operator of the form (4.1) with deficiency indices $(1, 1)$ and locally summable potential in $\mathcal{H} = L^2(\ell, \infty)$. Consider operator T_h of the form (4.2). Then*

1. *operator $\dot{A}$ has more than one non-negative self-adjoint extension, i.e., the Friedrichs extension A_F and the Kreĭn-von Neumann extension A_K do not coincide if and only if $m_\infty(-0) < \infty$;*
2. *operator T_h coincides with the Kreĭn-von Neumann extension if and only if $h = -m_\infty(-0)$;*
3. *operator T_h is accretive if and only if*

$$Re\, h \geq -m_\infty(-0); \tag{5.1}$$

4. *operator T_h, $(h \neq \bar{h})$ is α-sectorial if and only if $Re\, h > -m_\infty(-0)$ holds;*
5. *operator T_h, $(h \neq \bar{h})$ is accretive extremal if and only if $Re\, h = -m_\infty(-0)$;*
6. *if T_h, $(\mathrm{Im}\, h > 0)$ is α-sectorial, then the exact angle of sectoriality α can be calculated as*

$$\tan\alpha = \frac{\mathrm{Im}\, h}{Re\, h + m_\infty(-0)}. \tag{5.2}$$

It follows from item (2) of Theorem 11 that if $m_\infty(-0) = \infty$, then the operator T_h turns into a self-adjoint operator T_∞ corresponding to the Dirichlet problem

$$\begin{cases} T_\infty y = -y'' + q(x)y \\ y(\ell) = 0. \end{cases}$$

For the remainder of this paper we assume that $m_\infty(-0) < \infty$. Then according to Theorem 11 above (see also [24, 25]) we have the existence of the operator T_h, $(\mathrm{Im}\, h > 0)$ which is accretive and/or sectorial. The following was shown in [2]. Let T_h $(\mathrm{Im}\, h > 0)$ be an accretive Schrödinger operator of the form (4.2). Then for all real μ satisfying the inequality

$$\mu \geq \frac{(\mathrm{Im}\, h)^2}{m_\infty(-0) + \mathrm{Re}\, h} + \mathrm{Re}\, h, \tag{5.3}$$

the operators (4.3) form the set of all accretive $(*)$-extensions $\mathbb{A}$ of the operator T_h. The accretive operator T_h has a unique accretive $(*)$-extension $\mathbb{A}$ if and only if

$$\mathrm{Re}\, h = -m_\infty(-0).$$

In this case this unique $(*)$-extension (and its adjoint) has the form

$$\begin{aligned} \mathbb{A}y &= -y'' + q(x)y + [hy(\ell) - y'(\ell)]\,\delta(x-\ell), \\ \mathbb{A}^*y &= -y'' + q(x)y + [\overline{h}y(\ell) - y'(\ell)]\,\delta(x-\ell). \end{aligned} \tag{5.4}$$

Now suppose that $\Theta_{\mu,h}$ of the form (4.8) is an L-system with the main Schrödinger operator T_h defined by (4.2). We are going to tackle the Phillips-Kato extension problem with the help of the von Neumann parameter κ of T_h. Assume also that μ is given by (4.15) and hence T_h satisfies the Hypothesis 3 with real κ given by (4.14). Set

$$m_\infty(i) = A - iB,\;\; B > 0,\;\; m_\infty(-0) = m,\;\; C = (A-m)^2,\;\; D = C + B^2 > 0. \tag{5.5}$$

Then (4.14) yields

$$\begin{aligned} \kappa &= \left|\frac{m_\infty(i)+h}{m_\infty(i)+\overline{h}}\right| = \left|\frac{A - iB + \operatorname{Re} h + i\operatorname{Im} h}{A - iB + \operatorname{Re} h - i\operatorname{Im} h}\right| = \left|\frac{(A+\operatorname{Re} h) + i(\operatorname{Im} h - B)}{(A+\operatorname{Re} h) - i(\operatorname{Im} h + B)}\right| \\ &= \sqrt{\frac{(A+\operatorname{Re} h)^2 + (\operatorname{Im} h - B)^2}{(A+\operatorname{Re} h)^2 + (\operatorname{Im} h + B)^2}}, \end{aligned}$$

or

$$\kappa^2 = \frac{(A+\operatorname{Re} h)^2 + (\operatorname{Im} h - B)^2}{(A+\operatorname{Re} h)^2 + (\operatorname{Im} h + B)^2}. \tag{5.6}$$

Suppose that T_h is an extremal accretive operator. According to Theorem 11 we have that $\operatorname{Re} h = -m$. Applying this to (5.6) and using notations (5.5) gives

$$\begin{aligned} \kappa^2 &= \frac{(A-m)^2 + (\operatorname{Im} h - B)^2}{(A-m)^2 + (\operatorname{Im} h + B)^2} = \frac{C + (\operatorname{Im} h)^2 - 2B\operatorname{Im} h + B^2}{C + (\operatorname{Im} h)^2 + 2B\operatorname{Im} h + B^2} \\ &= \frac{(\operatorname{Im} h)^2 - 2B\operatorname{Im} h + D}{(\operatorname{Im} h)^2 + 2B\operatorname{Im} h + D}. \end{aligned}$$

Consider κ^2 as a function f of $H = \operatorname{Im} h$, that is

$$f(H) = \kappa^2(H) = \frac{H^2 - 2BH + D}{H^2 + 2BH + D}. \tag{5.7}$$

Taking the derivative of $f(H)$ in (5.7) and simplifying yields

$$f'(H) = \frac{4B(H^2 - D)}{(H^2 + BH + D)^2}.$$

Setting $f'(H) = 0$ and keeping in mind that $H = \operatorname{Im} h > 0$ and $D > 0$ we obtain a critical number

$$H = \sqrt{D},$$

that can be checked to be a point of minimum of $f(H)$ for all $H > 0$. Also, direct substitution gives

$$f(\sqrt{D}) = \frac{\sqrt{D} - B}{\sqrt{D} + B}.$$

Consequently,

$$\frac{\sqrt{D} - B}{\sqrt{D} + B} \le \kappa^2 < 1,$$

and hence

$$\sqrt{\frac{\sqrt{D} - B}{\sqrt{D} + B}} \le \kappa < 1,$$

or, after backward substitution and simplification,

$$\sqrt{\frac{\sqrt{|m_\infty(i)|^2 - 2m_\infty(-0)\mathrm{Re}\ m_\infty(i) + m_\infty^2(-0)} + \operatorname{Im} m_\infty(i)}{\sqrt{|m_\infty(i)|^2 - 2m_\infty(-0)\mathrm{Re}\ m_\infty(i) + m_\infty^2(-0)} - \operatorname{Im} m_\infty(i)}} \le \kappa < 1. \quad (5.8)$$

Thus, T_h is extremal accretive operator if and only if (5.8) holds, that is, the von Neumann parameter κ of T_h is such that $\kappa_0 \le \kappa < 1$, with

$$\kappa_0 = \sqrt{\frac{\sqrt{|m_\infty(i)|^2 - 2m_\infty(-0)\mathrm{Re}\ m_\infty(i) + m_\infty^2(-0)} + \operatorname{Im} m_\infty(i)}{\sqrt{|m_\infty(i)|^2 - 2m_\infty(-0)\mathrm{Re}\ m_\infty(i) + m_\infty^2(-0)} - \operatorname{Im} m_\infty(i)}}. \quad (5.9)$$

A sample graph of κ as a function of $\operatorname{Im} h$ is shown in Fig. 1.
We can summarize the above reasoning in the following theorem.

Theorem 12 *Under the condition (1) of Theorem 11, if a Schrödinger operator T_h of the form (4.2) with the modulus of von Neumann's parameter κ is extremal accretive, then $\kappa_0 \le \kappa < 1$, where κ_0 is given by (5.9).*

The proof follows from part (5) of Theorem 11 and formulas (5.8) and (5.9).
The following lemma contains a useful property of the function $m_\infty(z)$ that makes κ_0 (given by (5.9)) vanish.

Lemma 13 *Under the conditions of Theorem 12, the value of κ_0 given by (5.9) is zero if and only if*

$$Re\, m_\infty(i) = m_\infty(-0). \tag{5.10}$$

Proof Following our notation that we set in (5.5), from (5.9) we get that

$$\kappa_0 = \sqrt{\frac{\sqrt{D}-B}{\sqrt{D}+B}}. \tag{5.11}$$

Since $A = \mathrm{Re}\, m_\infty(i)$ and $m = m_\infty(-0)$, we need to show that $\kappa_0 = 0$ if and only if $A = m$.

If $A = m$, then using (5.5) we obtain

$$\sqrt{D} - B = \sqrt{C + B^2} - B = \sqrt{(A-m)^2 + B^2} - B = \sqrt{0 + B^2} - B = 0,$$

and hence the numerator of the fraction in (5.11) is zero making $\kappa_0 = 0$.

Conversely, assume that $\kappa_0 = 0$. Then $\sqrt{D} - B = 0$ and hence $D = B^2$. This implies $C = A - m = 0$ yielding (5.10). □

Now we will state and prove an inverse version of Theorem 12.

Theorem 14 *Let $\dot{A}$ be the same as in Theorems 11 and 12. If $\kappa_0 \le \kappa < 1$, with κ_0 given by (5.9), then there exist two (just one if $\kappa = \kappa_0$) extremal dissipative Schrödinger operators T_h, ($\dot{A} \subset T_h \subset \dot{A}^*$), of the form (4.2) such that their modulus of von Neumann's parameters equals κ.*

In this case, the boundary value(s) of h are given by

$$h = -m_\infty(-0) + \frac{B(1+\kappa^2) \pm \sqrt{4B^2\kappa^2 - C(1-\kappa^2)^2}}{1-\kappa^2}\, i, \tag{5.12}$$

where B and C are given by (5.5).

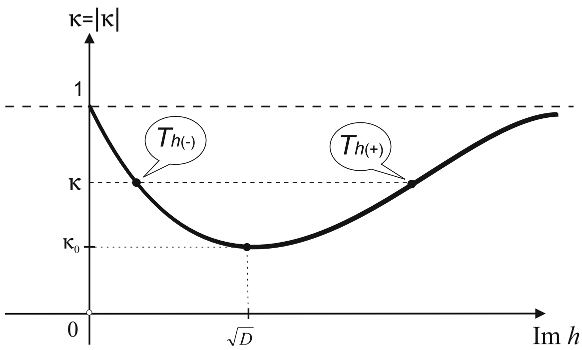

Fig. 1 Two extremal operators T_h with the same κ

Proof Let T_h be a Schrödinger operator of the form (4.2) with the same potential as in (4.1). All we need is to show that there exists a value of h that makes T_h an extremal dissipative quasi-self-adjoint extension of $\dot{A}$ whose modulus of von Neumann's parameter is equal to the given $\kappa \in [\kappa_0, 1)$. Note that equation (5.6) holds for any T_h with von Neumann's parameter κ. Setting $\operatorname{Re} h = -m$ in (5.6) will guarantee (see Theorem 11) that T_h is extremal and yields (5.7), that is,

$$\kappa^2 = \frac{x^2 - 2Bx + D}{x^2 + 2Bx + D}, \tag{5.13}$$

where $x = \operatorname{Im} h$. Modifying (5.13) leads to the quadratic equation

$$x^2 - 2B\left(\frac{1+\kappa^2}{1-\kappa^2}\right)x + D = 0. \tag{5.14}$$

We are going to show that equation (5.14) has at least one real positive solution. Setting

$$\xi = \frac{1+\kappa^2}{1-\kappa^2},$$

note that when $\kappa \in [\kappa_0, 1)$ we have $\xi \in [\xi_0, +\infty)$, where

$$\xi_0 = \frac{1 + \frac{\sqrt{D}-B}{\sqrt{D}+B}}{1 - \frac{\sqrt{D}-B}{\sqrt{D}+B}} = \frac{\sqrt{D}}{B}.$$

Consider the discriminant of the quadratic equation (5.14) as a function of ξ

$$f(\xi) = 4B^2\xi^2 - 4D.$$

Then its derivative $f'(\xi) = 8B^2\xi$ is always positive on $\xi \in [\xi_0, +\infty)$ indicating that $f(\xi)$ is an increasing function of $\xi \in [\xi_0, +\infty)$. Moreover, the direct check reveals that

$$f(\xi_0) = 4B^2\xi_0^2 - 4D = 0,$$

and hence $f(\xi)$ takes positive values at $\xi \in (\xi_0, +\infty)$. Applying the quadratic formula to equation (5.14) and taking into account that $B > 0$, $D > 0$ and $\xi > 0$, we obtain that

$$x = \frac{2B\xi \pm \sqrt{4B^2\xi^2 - 4D}}{2} = B\xi \pm \sqrt{B^2\xi^2 - D} \tag{5.15}$$

yields two positive real solutions. Therefore, the value of h such that $\operatorname{Re} h = -m_\infty(-0)$ and $\operatorname{Im} h$ equal to one of the values of x from (5.15) is the one that makes T_h

an extremal dissipative quasi-self-adjoint extension of $\dot{A}$. Substituting the expression for ξ into (5.15) and simplifying yields (5.12).

Clearly, if $\kappa \in (\kappa_0, 1)$, then (5.12) yields two distinct values of the boundary parameter h and hence there are two different Schrödinger operators T_h with von Neumann's parameter κ. In the case when $\kappa = \kappa_0$ the expression under the radical in (5.12) turns into zero and hence there is only one value of h yielding a unique extremal Schrödinger operator T_h with von Neumann's parameter $\kappa = \kappa_0$. □

Figure 1 gives good visual illustration of Theorem 14. There are two extremal operators $T_{h(-)}$ and $T_{h(+)}$ shown which share the same (modulus of) von Neumann's parameter κ. Here $h(-)$ and $h(+)$ are the values of the parameter h in (5.12) that correspond to the $(-)$ and $(+)$ signs of the formula.

6 On von Neumann's Parameter of Sectorial Schrödinger Operator T_h

The main goal of this section is (for a given $\beta \in (0, \pi/2)$) to explicitly describe the interval for the modulus of von Neumann's parameter κ of the operator T_h such that T_h is β-sectorial. Throughout this section we keep the assumption of the previous section that the symmetric operator $\dot{A}$ of the form (4.1) with deficiency indices (1,1) is nonnegative.

Consider T_h of the form (4.2) and assume that T_h is β-sectorial. Then according to Theorem 11 we have that $\operatorname{Re} h > -m_\infty(-0)$ and (see (5.2))

$$\cot\beta = \frac{\operatorname{Re} h + m_\infty(-0)}{\operatorname{Im} h},$$

or equivalently

$$\operatorname{Re} h + m_\infty(-0) = \cot\beta \cdot \operatorname{Im} h. \tag{6.1}$$

As in Sect. 5 we assume that Θ is an L-system of the form (4.8) with the main β-sectorial Schrödinger operator T_h. Suppose also that μ is given by (4.15) and hence T_h satisfies Hypothesis 3 with real κ given by (4.14). Once again, to simplify the calculation process we use our conventions described in (5.5). Combining (5.6) with (6.1) yields

$$\begin{aligned}
\kappa^2 &= \frac{(A + \operatorname{Re} h)^2 + (\operatorname{Im} h - B)^2}{(A + \operatorname{Re} h)^2 + (\operatorname{Im} h + B)^2} = \frac{(A - m + \cot\beta \cdot \operatorname{Im} h)^2 + (\operatorname{Im} h - B)^2}{(A - m + \cot\beta \cdot \operatorname{Im} h)^2 + (\operatorname{Im} h + B)^2} \\
&= \frac{(A-m)^2 + 2(A-m)\cot\beta \cdot \operatorname{Im} h + \cot^2\beta (\operatorname{Im} h)^2 + (\operatorname{Im} h)^2 - 2B \operatorname{Im} h + B^2}{(A-m)^2 + 2(A-m)\cot\beta \cdot \operatorname{Im} h + \cot^2\beta (\operatorname{Im} h)^2 + (\operatorname{Im} h)^2 + 2B \operatorname{Im} h + B^2} \\
&= \frac{(\cot^2\beta + 1)(\operatorname{Im} h)^2 + 2((A-m)\cot\beta - B)\operatorname{Im} h + (A-m)^2 + B^2}{(\cot^2\beta + 1)(\operatorname{Im} h)^2 + 2((A-m)\cot\beta + B)\operatorname{Im} h + (A-m)^2 + B^2} \\
&= \frac{(\operatorname{Im} h)^2 + 2\sin^2\beta((A-m)\cot\beta - B)\operatorname{Im} h + D\sin^2\beta}{(\operatorname{Im} h)^2 + 2\sin^2\beta((A-m)\cot\beta + B)\operatorname{Im} h + D\sin^2\beta}.
\end{aligned}$$

As in Sect. 5 we consider κ^2 as a function f of $H = \operatorname{Im} h$, that is

$$f(H) = \kappa^2(H) = \frac{H^2 + 2\sin^2\beta((A-m)\cot\beta - B)H + D\sin^2\beta}{H^2 + 2\sin^2\beta((A-m)\cot\beta + B)H + D\sin^2\beta}. \tag{6.2}$$

Taking the derivative of $f(H)$ in (6.2) we get

$$f'(H) = \frac{4\sin\beta\cdot B(H^2 - D\sin^2\beta)}{(H^2 + 2\sin^2\beta((A-m)\cot\beta + B)H + D\sin^2\beta)^2}.$$

Setting $f'(H) = 0$ and keeping in mind that $H = \operatorname{Im} h > 0$ and $D > 0$ we obtain a critical number

$$H = \sin\beta\cdot\sqrt{D},$$

that can be checked to be a point of minimum of $f(H)$ for all $H > 0$. Also, direct substitution gives

$$f(\sin\beta\sqrt{D}) = \frac{\sqrt{D} + (A-m)\cos\beta - B\sin\beta}{\sqrt{D} + (A-m)\cos\beta + B\sin\beta}.$$

Consequently,

$$\frac{\sqrt{D} + (A-m)\cos\beta - B\sin\beta}{\sqrt{D} + (A-m)\cos\beta + B\sin\beta} \le \kappa^2 < 1,$$

and hence

$$\sqrt{\frac{\sqrt{D} + (A-m)\cos\beta - B\sin\beta}{\sqrt{D} + (A-m)\cos\beta + B\sin\beta}} \le \kappa < 1,$$

or, after backward substitution and simplification, $\kappa_0 \le \kappa < 1$, where

$$\kappa_0 = \sqrt{\frac{\sqrt{D} + (A-m)\cos\beta - B\sin\beta}{\sqrt{D} + (A-m)\cos\beta + B\sin\beta}}. \tag{6.3}$$

In the above formula

$$\begin{aligned} \sqrt{D} &= \sqrt{|m_\infty(i)|^2 - 2m_\infty(-0)\operatorname{Re}\, m_\infty(i) + m_\infty^2(-0)}, \\ A - m &= \operatorname{Re}\, m_\infty(i) - m_\infty(-0), \qquad B = -\operatorname{Im}\, m_\infty(i). \end{aligned} \tag{6.4}$$

It is easy to see that when $\beta = \pi/2$, then (6.3) and (6.4) match the previous (extremal) case in the formula (5.9). A sample graph of κ as a function of $\operatorname{Im} h$ is shown in Fig. 2.

The following two theorems contain the main result of this section.

Theorem 15 *Under the condition (1) of Theorem 11, if a Schrödinger operator T_h of the form (4.2) with the modulus of von Neumann's parameter κ is β-sectorial ($\beta \in (0, \pi/2)$), then* $0 < \kappa_0 \le \kappa < 1$, *where κ_0 is given by* (6.3) *and* (6.4).

Proof The proof follows from part (4) of Theorem 11 and the reasoning of this section. We just note that $0 < \kappa_0$ because of the fact that a β-sectorial ($\beta \in (0, \pi/2)$) operator cannot have the von Neumann parameter equal to zero (see Lemma 2). □

Now we will state and prove an inverse version of Theorem 15 that is the analog of Theorem 14 from Sect. 5.

Theorem 16 *Let $\dot{A}$ be the same as in Theorem 15. Given $\beta \in (0, \pi/2)$ and $0 < \kappa_0 \le \kappa < 1$, where κ_0 is given by* (6.3), (6.4), *there exist two (one if $\kappa = \kappa_0$) β-sectorial dissipative Schrödinger operators T_h, ($\dot{A} \subset T_h \subset \dot{A}^*$), of the form* (4.2) *such that the modulus of their von Neumann's parameters equals κ. Moreover, the boundary value(s) of h are given by*

$$h = -m + \sin^2\beta(\cot\beta + i)\left(\xi B - (A - m)\cot\beta \pm \sqrt{E}\right), \tag{6.5}$$

where m, B and C are given by (5.5), $\xi = \frac{1+\kappa^2}{1-\kappa^2}$ *and*

$$E = [(A - m)\cot\beta - \xi B]^2 - D\csc^2\beta. \tag{6.6}$$

Proof We are going to use a similar to the proof of Theorem 14 method to confirm (6.5). We notice that the rational function in formula (6.2) resembles the one in (5.13). Therefore, the following formal change of coefficients

$$B_1 = \sin^2\beta((A - m)\cot\beta - B), \quad B_2 = \sin^2\beta((A - m)\cot\beta + B), \tag{6.7}$$

leads to

$$\kappa^2 = \frac{x^2 + 2B_1 x + D\sin^2\beta}{x^2 + 2B_2 x + D\sin^2\beta}, \tag{6.8}$$

where $x = \operatorname{Im} h$. Modifying (6.8) brings us to the quadratic equation

$$(\kappa^2 - 1)x^2 + 2(\kappa^2 B_2 - B_1)x + (\kappa^2 - 1)D\sin^2\beta = 0.$$

Taking into account that $\kappa^2 B_2 - B_1 = \sin^2\beta[(\kappa^2 - 1)(A - m)\cot\beta + (\kappa^2 + 1)B]$ we obtain

$$x^2 + 2\sin^2\beta\left[(A - m)\cot\beta + \frac{\kappa^2 + 1}{\kappa^2 - 1}B\right]x + D\sin^2\beta = 0.$$

Setting

$$\xi = \frac{1 + \kappa^2}{1 - \kappa^2},$$

note that when $\kappa \in [\kappa_0, 1)$ we have $\xi \in [\xi_0, +\infty)$, where (see (6.3))

$$\xi_0 = \frac{1+\kappa_0^2}{1-\kappa_0^2} = \frac{1+\frac{\sqrt{D}+(A-m)\cos\beta-B\sin\beta}{\sqrt{D}+(A-m)\cos\beta+B\sin\beta}}{1-\frac{\sqrt{D}+(A-m)\cos\beta-B\sin\beta}{\sqrt{D}+(A-m)\cos\beta+B\sin\beta}} = \frac{\sqrt{D}+(A-m)\cos\beta}{B\sin\beta}.$$

It is easy to see that if $\beta = \pi/2$, then the value of ξ_0 above matches the corresponding value of ξ_0 in the proof of Theorem 14 written for the extremal case. Furthermore, the above quadratic equation transforms into (see (6.7))

$$(\csc^2\beta)x^2 + 2\left[(A-m)\cot\beta - \xi B\right]x + D = 0. \tag{6.9}$$

Consider the discriminant of the quadratic equation (6.9) as a function of ξ

$$f(\xi) = 4\left[(A-m)\cot\beta - \xi B\right]^2 - 4D\csc^2\beta.$$

Then its derivative $f'(\xi) = 8B^2\xi - 8B(A-m)\cot\beta$ gives rise to a critical point $\xi_c = \frac{A-m}{B}\cot\beta$ that is clearly smaller than ξ_0, that is $\xi_c < \xi_0$. Consequently, $f'(\xi)$ is always positive on $\xi \in [\xi_0, +\infty)$ indicating that $f(\xi)$ is an increasing function of $\xi \in [\xi_0, +\infty)$. Moreover, the direct check reveals that

$$f(\xi_0) = 4[(A-m)\cot\beta - \xi_0 B]^2 - 4D(\csc^2\beta) = 4\left[\frac{-\sqrt{D}}{\sin\beta}\right]^2 - 4D\csc^2\beta = 0,$$

and hence $f(\xi)$ takes positive values on $\xi \in (\xi_0, +\infty)$. Applying the quadratic formula to equation (6.9) and taking into account that $B > 0$, $D > 0$ and $\xi > 0$, we obtain that

$$\begin{aligned} x &= \frac{2\left[\xi B - (A-m)\cot\beta\right] \pm \sqrt{4\left[(A-m)\cot\beta - \xi B\right]^2 - 4D\csc^2\beta}}{2\csc^2\beta} \\ &= \sin^2\beta\left(\xi B - (A-m)\cot\beta \pm \sqrt{\left[(A-m)\cot\beta - \xi B\right]^2 - D\csc^2\beta}\right) \\ &= \sin^2\beta\left(\xi B - (A-m)\cot\beta \pm \sqrt{E}\right) \end{aligned} \tag{6.10}$$

yields two positive real solutions. Therefore, the value of h such that $\operatorname{Re} h = -m_\infty(-0) + (\cot\beta)x$, where $x = \operatorname{Im} h$ is given by (6.10), defines β-sectorial dissipative Schrödinger operators T_h. Substituting the expression for ξ into (6.10) and simplifying yields (6.5).

It can be clearly seen from (6.5) that if $\kappa \in (\kappa_0, 1)$, then there are two distinct values of the boundary parameter h and hence two different β-sectorial Schrödinger operators T_h with von Neumann's parameter. In the case when $\kappa = \kappa_0$ there is only one β-sectorial Schrödinger operators T_h with this property. □

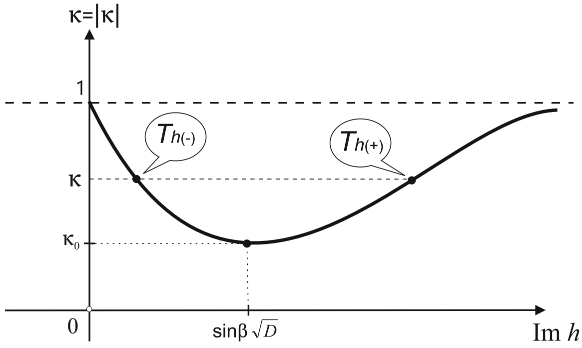

Fig. 2 Two β-sectorial operators T_h with the same κ

Figure 2 gives good visualization of Theorem 16. It depicts two β-sectorial operators $T_{h(-)}$ and $T_{h(+)}$ shown which share the same (modulus of) von Neumann's parameter κ. Here $h(-)$ and $h(+)$ are the values of the parameter h in (6.5) that correspond to the $(-)$ and $(+)$ signs of the formula.

We finalize the section with a lemma that contains a very useful property of the function $m_\infty(z)$ related to a nonnegative symmetric operator $\dot{A}$ of the form (4.1).

Lemma 17 *Let $\dot{A}$ be a nonnegative symmetric Schrödinger operator of the form* (4.1) *with deficiency indices* (1, 1) *and locally summable potential in $\mathcal{H} = L^2(\ell, \infty)$. If the corresponding Weyl function $m_\infty(z)$ has the property $m_\infty(-0) < \infty$, then*

$$Re\, m_\infty(i) \geq m_\infty(-0). \tag{6.11}$$

Proof We stick to the notation that we set in (5.5), that is $m_\infty(i) = A - iB$, $B > 0$, $m_\infty(-0) = m$, $C = (A - m)^2$, and $D = C + B^2 > 0$. We need to show that $A \geq m$. Assume the contrary, $A < m$. Since $m < \infty$, for every $\beta \in (0, \pi/2)$ there exists a β-sectorial operator T_h of the form (4.2) (see [2, 23]). Take a β_0 such that

$$\cos\beta_0 = \frac{m - A}{\sqrt{D}} \quad \text{or} \quad \beta_0 = \arccos\left(\frac{m - A}{\sqrt{D}}\right). \tag{6.12}$$

Since under our assumption $A < m$, then $\beta_0 \in (0, \pi/2)$ and hence there is a β_0-sectorial operator T_h of the form (4.2). Moreover, according to Theorem 16 the boundary value h of this T_h is determined by (6.5) and its von Neumann's parameter is κ_0 given by (6.3). Using the basic trig identity and (5.5) we get

$$\sin\beta_0 = \sqrt{1 - \cos^2\beta_0} = \sqrt{1 - \frac{C}{D}} = \frac{\sqrt{D - C}}{\sqrt{D}} = \frac{\sqrt{B^2}}{\sqrt{D}} = \frac{B}{\sqrt{D}}. \tag{6.13}$$

Substituting $\cos\beta_0$ and $\sin\beta_0$ from (6.12) and (6.13) into (6.3) reveals

$$\kappa_0^2 = \frac{\sqrt{D} - \sqrt{C}\cdot\frac{\sqrt{C}}{\sqrt{D}} - B\cdot\frac{B}{\sqrt{D}}}{\sqrt{D} - \sqrt{C}\cdot\frac{\sqrt{C}}{\sqrt{D}} + B\cdot\frac{B}{\sqrt{D}}} = \frac{D - C - B^2}{D + B^2 - C} = \frac{D - D}{D + B^2 - C} = 0,$$

or $\kappa_0 = 0$. Hence under our assumption we have the existence of a β_0-sectorial operator T_h with von Neumann's parameter $\kappa_0 = 0$. This contradicts Lemma 2 and thus our assumption that $A < m$ is wrong. Consequently, $A \geq m$. □

7 c-Entropy of an L-System and Minimal Dissipation Coefficient

We begin with introducing a concept of the c-entropy of an L-system.

Definition 18 Let Θ be an L-system of the form (3.10). The quantity

$$\mathcal{S} = -\ln(|W_\Theta(-i)|), \tag{7.1}$$

where $W_\Theta(z)$ is the transfer function of Θ, is called the **coupling entropy** (or **c-entropy**) of the L-system Θ.

There is an alternative operator-theoretic way to define the c-entropy. If T is the main operator of the L-system Θ and κ is von Neumann's parameter of T in some basis $g_\pm$, then (see [7]) $|W_\Theta(-i)| = |\kappa|$ and hence

$$\mathcal{S} = -\ln(|W_\Theta(-i)|) = -\ln(|\kappa|). \tag{7.2}$$

We emphasize that c-entropy defined by formula (7.2) does not depend on the choice of deficiency basis $g_\pm$ and moreover is an additive function with respect to the coupling of L-systems (see [7]). Note that if, in addition, the point $z = i$ belongs to $\rho(T)$, then we also have that

$$\mathcal{S} = \ln(|W_\Theta(i)|) = \ln(1/|\kappa|) = -\ln(|\kappa|). \tag{7.3}$$

This follows from the known (see [2]) property of the transfer functions for L-systems that states that $W_\Theta(z)\overline{W_\Theta(\bar{z})} = 1$ and the fact that $|W_\Theta(i)| = 1/|\kappa|$ (see [6]).

Next we pose for the following optimization problem associated with the c-entropy of Schrö-inger L-systems:

- *Describe L-systems with the Schrödinger operator with a given c-entropy and minimal dissipation coefficient.*

The main goal of this section is to provide a solution of the problem above for the L-systems whose impedance functions belong to the (generalized) Donoghue classes $\mathfrak{M}$, $\mathfrak{M}_\kappa$, and $\mathfrak{M}_\kappa^{-1}$.

Let $\Theta_{\mu,h}$ be a Schrödinger L-system one of (4.8) based upon the symmetric operator $\dot{A}$ of the form (4.1) with corresponding Weyl function $m_\infty(z)$. Also, let κ denote the modulus of the von Neumann parameter of the main operator T_h of the L-system $\Theta_{\mu,h}$. Suppose that the impedance function $V_{\Theta_{\mu,h}}(z)$ given by (4.10), belongs to a generalized Donoghue class. Then

$$V_{\Theta_{\mu,h}}(i) = ai, \tag{7.4}$$

for some $a > 0$. Representation (7.4) has been justified in the proof of Theorem 9 for $0 < a < 1$ (see (4.25)) and can be verified for $a > 1$ (cf. Theorem 10). Depending on the value of a in (7.4) we (see (3.6) and (3.7)) have

$$\kappa = \begin{cases} \frac{1-a}{1+a}, & 0 < a < 1; \\ 0, & a = 1; \\ \frac{a-1}{1+a}, & a > 1. \end{cases} \tag{7.5}$$

In order to address the problem above we point out that in accordance with (7.2) the knowledge of c-entropy fixes the value of $\kappa = |\kappa|$ and hence of a as well. Moreover, by Theorem 9 for the coefficient of dissipation $\mathrm{Im}\, h$ we have the explicit representation

$$\mathrm{Im}\, h(\mu) = -\frac{ad^3 + ad(c+\mu)^2}{a^2d^2 + (c+\mu)^2}, \tag{7.6}$$

where $c = \mathrm{Re}\, m_\infty(i)$, $d = \mathrm{Im}\, m_\infty(i)$, and μ is the parameter that determines the L-system $\Theta_{\mu,h}$. Thus, to minimize $\mathrm{Im}\, h$ for a given $a > 0$ we seek the corresponding critical value of μ. Taking the derivative of $\mathrm{Im}\, h$ as a function of μ, simplifying, and setting it equal to zero yields

$$\frac{d}{d\mu}(\mathrm{Im}\, h) = \frac{2ad^3(c+\mu)(1-a^2)}{(a^2d^2 + (c+\mu)^2)^2} = 0.$$

Therefore,

$$\mu = -c = -\mathrm{Re}\, m_\infty(i)$$

is the only critical point of $\mathrm{Im}\, h(\mu)$. It is easy to check that this critical value of μ gives the minimum for $\mathrm{Im}\, h$ whenever $a > 1$ and the maximum whenever $0 < a < 1$. Also, clearly $\mathrm{Im}\, h \equiv -d = -\mathrm{Im}\, m_\infty(i)$ if $a = 1$. Moreover, as we have shown in the proof of Theorem 9, in this case

$$\mathrm{Im}\, h = -(1/a)\, \mathrm{Im}\, m_\infty(i). \tag{7.7}$$

This reasoning leads to the following result.

Theorem 19 *Assume that an L-system $\Theta_{\mu,h}$ of the form (4.8) with the main operator T_h has c-entropy $\mathcal{S}$. Then, under the constraint that the impedance function $V_{\Theta_{\mu,h}}(z)$ belongs to the generalized Donoghue class $\mathfrak{M}_\kappa^{-1}$ with $\kappa = e^{-\mathcal{S}}$, the L-system $\Theta_{\mu,h}$ has the minimum coefficient of dissipation*

$$\mathcal{D}_{min} = \mathrm{Im}\, h = -\left(\frac{e^{\mathcal{S}}-1}{e^{\mathcal{S}}+1}\right) \mathrm{Im}\, m_\infty(i) = -\tanh\left(\frac{\mathcal{S}}{2}\right) \mathrm{Im}\, m_\infty(i). \tag{7.8}$$

In this case $\Theta_{\mu,h}$ is uniquely determined by the parameters

$$h = -Re\, m_\infty(i) - i \tanh\left(\frac{\mathcal{S}}{2}\right) \operatorname{Im} m_\infty(i) \quad and \quad \mu = -Re\, m_\infty(i). \tag{7.9}$$

Proof Since out L-system $\Theta_{\mu,h}$ has a fixed given c-entropy $\mathcal{S}$, then the modulus of von Neumann's parameter κ of its main operator T_h is given by $\kappa = e^{-\mathcal{S}}$. Moreover, since $V_{\Theta_{\mu,h}}(z) \in \mathfrak{M}_\kappa^{-1}$, then $V_{\Theta_{\mu,h}}(i) = ai$ for some $a > 1$. Using this value of κ and formula (7.5) we obtain

$$a = \frac{1+\kappa}{1-\kappa} > 1.$$

We have

$$\frac{1}{a} = \frac{1-e^{-\mathcal{S}}}{1+e^{-\mathcal{S}}} = \frac{e^{\mathcal{S}}-1}{e^{\mathcal{S}}+1} = \tanh\left(\frac{\mathcal{S}}{2}\right).$$

The rest follows from the fact that $\mu = -c = -\operatorname{Re} m_\infty(i)$ is the point of minimum of $\operatorname{Im} h$ if $a > 1$, formulas (4.22) (in the context of Theorem 10) and (7.7). □

The L-system $\Theta_{\mu,h}$ with the minimum coefficient of dissipation in Theorem 19 can be written explicitly. In particular its state space operator $\mathbb{A}_{\mu,h}$ can be described by formulas (4.3) with the values of h and μ given by (7.9). That is,

$$\begin{aligned}
\mathbb{A}_{\mu,h}\, y &= -y'' + q(x)y + \frac{y'(\ell) - hy(\ell)}{\operatorname{Re} m_\infty(i) + h}\,[\delta'(x-\ell) - \operatorname{Re} m_\infty(i)\delta(x-\ell)],\\
\mathbb{A}^*_{\mu,h}\, y &= -y'' + q(x)y + \frac{y'(\ell) - \overline{h}y(\ell)}{\operatorname{Re} m_\infty(i) + \overline{h}}\,[\delta'(x-\ell) - \operatorname{Re} m_\infty(i)\delta(x-\ell)],
\end{aligned} \tag{7.10}$$

where

$$h = -\operatorname{Re} m_\infty(i) - i \tanh\left(\frac{\mathcal{S}}{2}\right) \operatorname{Im} m_\infty(i).$$

An analogues result similar to Theorem 19 takes place whenever $0 < a < 1$.

Theorem 20 *Assume that an L-system $\Theta_{\mu,h}$ of the form (4.8) with a main operator T_h has c-entropy $\mathcal{S}$. Then, under the constraint that the impedance function $V_{\Theta_{\mu,h}}(z)$ belongs to the generalized Donoghue class $\mathfrak{M}_\kappa$ with $\kappa = e^{-\mathcal{S}}$, the L-system $\Theta_{\mu,h}$ has the minimum coefficient of dissipation*

$$\mathcal{D}_{min} = -\tanh\left(\frac{\mathcal{S}}{2}\right) \operatorname{Im} m_\infty(i).$$

In this case, $\Theta_{\mu,h}$ is uniquely determined by the parameters

$$h = -Re\, m_\infty(i) - i \tanh\left(\frac{\mathcal{S}}{2}\right) \operatorname{Im} m_\infty(i) \quad and \quad \mu = \infty. \tag{7.11}$$

Proof Since $V_{\Theta_{\mu,h}}(z) \in \mathfrak{M}_\kappa$, we have $V_{\Theta_{\mu,h}}(i) = ai$ for some $0 < a < 1$. Hence by (7.5), since $0 < a < 1$, we have $\kappa = e^{-\mathcal{S}} = \frac{1-a}{1+a}$. It was shown in [2] that there exists a unique Schrödinger L-system $\Theta_{\tilde{\mu},h}$ of the form (4.8) with the same main operator T_h as in $\Theta_{\mu,h}$ such that $V_{\Theta_{\tilde{\mu},h}}(z) = -1/V_{\Theta_{\mu,h}}(z)$. Indeed, this L-system $\Theta_{\tilde{\mu},h}$ is determined via (4.3) by the parameters h and

$$\tilde{\mu} = \frac{\mu \operatorname{Re} h - |h|^2}{\mu - \operatorname{Re} h}. \tag{7.12}$$

Furthermore, $V_{\Theta_{\tilde{\mu},h}}(i) = \tilde{a}i = -1/V_{\Theta_{\mu,h}}(i) = i/a$ and hence $\tilde{a} = 1/a > 1$. Both L-systems $\Theta_{\mu,h}$ and $\Theta_{\tilde{\mu},h}$ share the same main operator T_h and thus the same c-entropy $\mathcal{S}$. Repeating the argument in the proof of Theorem 19 we obtain

$$\tilde{a} = \frac{1+\kappa}{1-\kappa} > 1 \quad \text{and} \quad \frac{1}{\tilde{a}} = \frac{1}{1/a} = \frac{e^{\mathcal{S}} - 1}{e^{\mathcal{S}} + 1} = \tanh\left(\frac{\mathcal{S}}{2}\right).$$

Applying (7.7) yields

$$\begin{aligned} \operatorname{Im} h = \mathcal{D}_{min} &= -\frac{1}{\tilde{a}} \operatorname{Im} m_\infty(i) = -\left(\frac{e^{\mathcal{S}} - 1}{e^{\mathcal{S}} + 1}\right) \operatorname{Im} m_\infty(i) \\ &= -\tanh\left(\frac{\mathcal{S}}{2}\right) \operatorname{Im} m_\infty(i). \end{aligned}$$

To find Re h in this case we substitute $\tilde{\mu} = -c = -\operatorname{Re} m_\infty(i)$ in (4.23). In order to obtain the value of μ that defines $\Theta_{\mu,h}$ we set $\tilde{\mu} = -c$ in (7.12) and solve for μ. We have

$$-c = \frac{\mu \operatorname{Re} h - |h|^2}{\mu - \operatorname{Re} h},$$

yielding

$$\mu = \frac{c \operatorname{Re} h + |h|^2}{c + \operatorname{Re} h} = \infty.$$

Thus (7.11) takes place and the proof is complete. □

We note that if $a = 1$ in the above consideration, then the c-entropy is infinite, $\mathcal{S} = +\infty$, and $\kappa = 0$ implying

$$\operatorname{Im} h = -\operatorname{Im} m_\infty(i)$$

(see Theorem 8).

As before, the L-system $\Theta_{\mu,h}$ with the given c-entropy and minimum coefficient of dissipation in the statement of Theorem 20 can be written explicitly. In particular, its state space operator $\mathbb{A}$ is described by formulas (4.3) with the values of h given by (7.11), and $\mu = \infty$ as follows:

$$\begin{aligned} \mathbb{A}_{\infty,h}\, y &= -y'' + q(x)y - [y'(\ell) - hy(\ell)]\,\delta(x-\ell), \\ \mathbb{A}^*_{\infty,h}\, y &= -y'' + q(x)y - [y'(\ell) - \overline{h}\, y(\ell)]\,\delta(x-\ell), \end{aligned} \tag{7.13}$$

where

$$h = -\operatorname{Re} m_\infty(i) - i \tanh\left(\frac{\mathcal{S}}{2}\right) \operatorname{Im} m_\infty(i).$$

8 L-Systems with Schrödinger's Operator and Maximal c-Entropy

In this section we set focus on the second part of the dual c-entropy problem that was announced in Introduction, that is:

- *Describe an L-system with the Schrödinger main operator with a given dissipation coefficient and the maximal c-entropy.*

Again we are going to consider L-systems with Schrödinger operator whose impedance functions belong to the generalized Donoghue classes $\mathfrak{M}$, $\mathfrak{M}_\kappa$, and $\mathfrak{M}_\kappa^{-1}$. Within this class of L-systems we will describe the ones that have a given dissipation coefficient and the *maximal finite* c-entropy.

Let $\Theta_{\mu,h}$ be a Schrödinger L-system of the form (4.8). Let also κ be the von Neumann parameter of the main operator T_h in $\Theta_{\mu,h}$. Suppose $V_{\Theta_{\mu,h}}(z)$ is the impedance function of (4.10) that belongs to one of the generalized Donoghue classes. Then, as we have shown in the proof of Theorem 9, the relation $V_{\Theta_{\mu,h}}(i) = ai$ is given by (4.25) for some $a > 0$. Depending on the value of a, the parameter κ is related to a via (7.5).

To address the problem of maximizing the c-entropy using a given dissipation coefficient we start at relation (4.25). Recall the notations we set in (4.26), that is $c = \operatorname{Re} m_\infty(i)$, $d = \operatorname{Im} m_\infty(i)$, $x = \operatorname{Re} h$, $y = \operatorname{Im} h$. As we have shown in the proof of Theorem 9 relation (4.25) leads to (4.27) which (after equating real and imaginary parts on both sides of (4.27)) yields

$$\begin{aligned} (c+\mu)y &= ad(x-\mu), \\ dy &= ac(\mu - x) + a\mu x - ax^2 - ay^2. \end{aligned} \tag{8.1}$$

Solving the first equation in (8.1) gives us

$$x - \mu = \frac{(c+\mu)y}{ad} \quad \text{and} \quad \mu = \frac{adx - cy}{ad + y}. \tag{8.2}$$

Substituting the expressions from (8.2) into the second equation (8.1) yields

$$dy = -ac \cdot \frac{(c+\mu)y}{ad} - ax \cdot \frac{(c+\mu)y}{ad} - ay^2,$$

or (after cancelation)

$$d^2 = -c(c+\mu) - x(c+\mu) - ady.$$

Solving for a results in

$$a = -\frac{(c+\mu)(c+x) + d^2}{dy}.$$

Plugging the expression for μ from (8.2) in the above and simplifying leads us to the quadratic equation for a

$$dy\, a^2 + [(c+x)^2 + y^2 + d^2]\, a + dy = 0. \tag{8.3}$$

Notice the discriminant D of the quadratic equation (8.3) is non-negative and equals zero if only if $c = -x$ and $y = -d$. Indeed,

$$\begin{aligned} D &= [(c+x)^2 + y^2 + d^2]^2 - 4d^2y^2 \\ &= ((c+x)^2 + y^2 + d^2 - 2dy)((c+x)^2 + y^2 + d^2 + 2dy) \\ &= \left((c+x)^2 + (y-d)^2\right)\left((c+x)^2 + (y+d)^2\right) \geq 0. \end{aligned}$$

Therefore, if $c \neq -x$ or $y \neq -d$ equation (8.3) has two distinct real roots

$$a_{1,2} = \frac{-(c+x)^2 - y^2 - d^2 \pm \sqrt{[(c+x)^2 + y^2 + d^2]^2 - 4d^2y^2}}{2dy}. \tag{8.4}$$

Set

$$\nu = (c+x)^2 + y^2 + d^2. \tag{8.5}$$

Our goal is to show that

$$a_1 = \frac{-\nu + \sqrt{\nu^2 - 4d^2y^2}}{2dy} < 1.$$

Since $d < 0$ and $y > 0$, the above inequality is equivalent to

$$-\nu + \sqrt{\nu^2 - 4d^2y^2} > 2dy \quad \text{or} \quad \nu^2 - 4d^2y^2 > (\nu + 2dy)^2,$$

which transforms to

$$(\nu + 2dy)^2 - (\nu + 2dy)(\nu - 2dy) < 0 \quad \text{or} \quad 4dy(\nu + 2dy) < 0.$$

Taking into account that $4dy < 0$, we get

$$\nu + 2dy = (c + x)^2 + y^2 + d^2 = (c + x)^2 + (y + 2)^2 > 0,$$

for $c \neq -x$ or $y \neq -d$. Reversing the argument and following the chain of equivalent inequalities backward we obtain

$$a_1 = \frac{-(c + x)^2 - y^2 - d^2 + \sqrt{[(c + x)^2 + y^2 + d^2]^2 - 4d^2y^2}}{2dy} < 1, \tag{8.6}$$

for $c \neq -x$ or $y \neq -d$.

Our next goal is to show that

$$a_2 = \frac{-\nu - \sqrt{\nu^2 - 4d^2y^2}}{2dy} > 1,$$

for $c \neq -x$ or $y \neq -d$. The above inequality is equivalent to

$$-\nu - \sqrt{\nu^2 - 4d^2y^2} < 2dy,$$

which transforms to

$$\sqrt{\nu^2 - 4d^2y^2} > -\nu - 2dy = -(c + x)^2 - y^2 - d^2 - 2dy = -(c + x)^2 - (y + d)^2.$$

The last inequality is always true for any $c \neq -x$ or $y \neq -d$ (we are comparing a positive number on the left to a negative number on the right). As before, we pull the chain of equivalent inequalities backward to obtain

$$a_2 = \frac{-(c + x)^2 - y^2 - d^2 - \sqrt{[(c + x)^2 + y^2 + d^2]^2 - 4d^2y^2}}{2dy} > 1, \tag{8.7}$$

for $c \neq -x$ or $y \neq -d$.

Finally we observe that substituting $c = -x$ and $y = -d$ into (8.4) results in

$$a = a_1 = a_2 = 1, \quad c = -x, \quad y = -d. \tag{8.8}$$

Now we are ready to state the following.

Theorem 21 *Assume that an L-system $\Theta_{\mu,h}$ of the form (4.8) with the main operator T_h and coefficient of dissipation $\mathcal{D} = \operatorname{Im} h \neq -\operatorname{Im} m_\infty(i)$. Then, under the constraint* *that the impedance function $V_{\Theta_{\mu,h}}(z) \in \mathfrak{M}_\kappa$ for some $0 < \kappa < 1$, the L-system $\Theta_{\mu,h}$ has the maximum finite c-entropy $\mathcal{S}_{max}$ if and only if it is determined by parameters*

$$h = -Re\, m_\infty(i) + i\mathcal{D} \quad and \quad \mu = \begin{cases} -Re\, m_\infty(i), & if\, \mathcal{D} > -\operatorname{Im} m_\infty(i); \\ \infty, & if\, \mathcal{D} < -\operatorname{Im} m_\infty(i). \end{cases} \tag{8.9}$$

In this case, $\mathcal{S}_{max}$ is determined by the formula

$$\mathcal{S}_{max} = \ln|\mathcal{D} - \operatorname{Im} m_\infty(i)| - \ln|\mathcal{D} + \operatorname{Im} m_\infty(i)|. \tag{8.10}$$

Proof Since $V_{\Theta_{\mu,h}}(z) \in \mathfrak{M}_\kappa$, then it satisfies a normalization condition

$$V_{\Theta_{\mu,h}}(i) = ai \quad \text{for} \quad a = \frac{1-\kappa}{1+\kappa}.$$

Moreover, since $0 < a < 1$ in the above, then formula (8.6) takes place and hence

$$a(x) = \frac{-(c+x)^2 - \mathcal{D}^2 - d^2 + \sqrt{[(c+x)^2 + \mathcal{D}^2 + d^2]^2 - 4d^2\mathcal{D}^2}}{2d\mathcal{D}}. \tag{8.11}$$

Taking the derivative of $a(x)$ in (8.11) and setting it equal to zero yields

$$\begin{aligned} a'(x) &= \frac{c+x}{d\mathcal{D}} \left(\frac{(c+x)^2 + \mathcal{D}^2 + d^2}{\sqrt{[(c+x)^2 + \mathcal{D}^2 + d^2]^2 - 4d^2\mathcal{D}^2}} - 1 \right) \\ &= \frac{c+x}{d\mathcal{D}} \cdot \frac{\nu - \sqrt{\nu^2 - 4d^2\mathcal{D}^2}}{\sqrt{\nu^2 - 4d^2\mathcal{D}^2}} = (c+x) \cdot \frac{\nu - \sqrt{\nu^2 - 4d^2\mathcal{D}^2}}{d\mathcal{D}\sqrt{\nu^2 - 4d^2\mathcal{D}^2}} = 0, \end{aligned}$$

where ν is defined in (8.5). By inspection,

$$\frac{\nu - \sqrt{\nu^2 - 4d^2\mathcal{D}^2}}{d\mathcal{D}\sqrt{\nu^2 - 4d^2\mathcal{D}^2}} < 0$$

and that makes the only critical number $x = -c$ to be a point of maximum. Moreover, this maximum value of a is

$$a(-c) = \begin{cases} -d/\mathcal{D}, \text{ if } \mathcal{D} > -d; \\ -\mathcal{D}/d, \text{ if } \mathcal{D} < -d. \end{cases} \tag{8.12}$$

This follows from the direct substitution of $x = -c$ into (8.11) giving

$$\begin{aligned} a(-c) &= \frac{-\mathcal{D}^2 - d^2 + \sqrt{[\mathcal{D}^2 + d^2]^2 - 4d^2\mathcal{D}^2}}{2d\mathcal{D}} = \frac{-\mathcal{D}^2 - d^2 + \sqrt{[\mathcal{D}^2 - d^2]^2}}{2d\mathcal{D}} \\ &= \frac{-\mathcal{D}^2 - d^2 + |\mathcal{D}^2 - d^2|}{2d\mathcal{D}}. \end{aligned}$$

Suppose that our L-system $\Theta_{\mu,h}$ with a given coefficient of dissipation $\mathcal{D} = \operatorname{Im} h \neq \operatorname{Im} m_\infty(i)$ has the maximum finite c-entropy $\mathcal{S}_{max}$. It follows from the definition of c-entropy and formula (7.2) that the maximum finite c-entropy is achieved when the modulus of von Neumann parameter κ is at its minimum. Since $0 < a < 1$, the corresponding κ is related to a via (3.6). Consider κ as a function of a single variable x. Then

$$\kappa(x) = \frac{1 - a(x)}{1 + a(x)}$$

is a decreasing function of a and hence $\kappa(x)$ attains its minimum whenever $a(x)$ assumes its maximum. But as we have just shown above $a(x)$ has its absolute maximum at $x = -c$. Therefore, the L-system $\Theta_{\mu,h}$ has the maximum c-entropy $\mathcal{S}_{max}$ when $x = -c = -\operatorname{Re} m_\infty(i)$. The corresponding values of system parameters h and μ are found once we recall that $x = \operatorname{Re} h$ and $\mathcal{D} = \operatorname{Im} h$ and then substitute $x = -c = -\operatorname{Re} m_\infty(i)$ in (8.2) considering both cases for a in (8.12). This proves the necessity part of the statement.

Conversely, let the L-system $\Theta_{\mu,h}$ has parameters h and μ defined by (8.9). Then as it follows from the first part of (8.9) we have $x = -c = -\operatorname{Re} m_\infty(i)$. As we have shown in the first part of the proof, this value of $x = -c$ maximizes $a(x)$, minimizes $\kappa(x)$, and hence maximizes the c-entropy of our L-system. Thus, $\Theta_{\mu,h}$ has the maximum finite c-entropy $\mathcal{S}_{max}$ if (8.9) holds.

In order to prove (8.10) we observe that

$$\kappa(-c) = \begin{cases} \frac{\mathcal{D}+d}{\mathcal{D}-d}, & \text{if } \mathcal{D} > -d; \\ \frac{\mathcal{D}+d}{d-\mathcal{D}}, & \text{if } \mathcal{D} < -d, \end{cases} \tag{8.13}$$

is the absolute minimum of κ for all real x. It follows from the definition of c-entropy and formula (7.2) that the minimum of κ corresponds to the maximum of c-entropy and we have

$$\mathcal{S}(-c) = -\ln|\kappa(-c)| = -\ln\left|\frac{\mathcal{D}+d}{\mathcal{D}-d}\right| = \ln|\mathcal{D}-d| - \ln|\mathcal{D}+d|,$$

confirming (8.10). □

A similar result takes place when $a > 1$.

Theorem 22 *Assume that an L-system $\Theta_{\mu,h}$ of the form (4.8) with the main operator T_h and coefficient of dissipation $\mathcal{D} = \operatorname{Im} h \neq -\operatorname{Im} m_\infty(i)$. Then, under constraint that the impedance function $V_{\Theta_{\mu,h}}(z) \in \mathfrak{M}_\kappa^{-1}$ for some $0 < \kappa < 1$, the L-system $\Theta_{\mu,h}$ has the maximum finite c-entropy $\mathcal{S}_{max}$ if and only if it is determined by parameters*

$$h = -Re\, m_\infty(i) + i\mathcal{D} \quad and \quad \mu = \begin{cases} \infty, & if\, \mathcal{D} > -\operatorname{Im} m_\infty(i); \\ -Re\, m_\infty(i), & if\, \mathcal{D} < -\operatorname{Im} m_\infty(i). \end{cases} \tag{8.14}$$

In this case, $\mathcal{S}_{max}$ is determined by (8.10).

Proof Since $V_{\Theta_{\mu,h}}(z) \in \mathfrak{M}_\kappa^{-1}$, it satisfies a normalization condition

$$V_{\Theta_{\mu,h}}(i) = ai \quad \text{for} \quad a = \frac{1+\kappa}{1-\kappa}.$$

Also, since $a > 1$ in the above, then formula (8.7) takes place and hence

$$a(x) = \frac{-(c+x)^2 - \mathcal{D}^2 - d^2 - \sqrt{[(c+x)^2 + \mathcal{D}^2 + d^2]^2 - 4d^2\mathcal{D}^2}}{2d\mathcal{D}}. \tag{8.15}$$

Taking the derivative of $a(x)$ in (8.15) and setting it equal to zero yields

$$\begin{aligned} a'(x) &= -\frac{c+x}{d\mathcal{D}}\left(\frac{(c+x)^2 + \mathcal{D}^2 + d^2}{\sqrt{[(c+x)^2 + \mathcal{D}^2 + d^2]^2 - 4d^2\mathcal{D}^2}} + 1\right) \\ &= -\frac{c+x}{d\mathcal{D}} \cdot \frac{\nu + \sqrt{\nu^2 - 4d^2\mathcal{D}^2}}{\sqrt{\nu^2 - 4d^2\mathcal{D}^2}} = -(c+x) \cdot \frac{\nu + \sqrt{\nu^2 - 4d^2\mathcal{D}^2}}{d\mathcal{D}\sqrt{\nu^2 - 4d^2\mathcal{D}^2}} = 0, \end{aligned}$$

where ν is defined in (8.5). Clearly, we have that

$$\frac{\nu - \sqrt{\nu^2 - 4d^2\mathcal{D}^2}}{d\mathcal{D}\sqrt{\nu^2 - 4d^2\mathcal{D}^2}} > 0$$

and $a(x)$ has the only critical number $x = -c$ a point of minimum. Moreover, the minimum value of a is

$$a(-c) = \begin{cases} -\mathcal{D}/d, & \text{if } \mathcal{D} > -d; \\ -d/\mathcal{D}, & \text{if } \mathcal{D} < -d. \end{cases} \tag{8.16}$$

This follows from the direct substitution of $x = -c$ into (8.11) giving

$$\begin{aligned} a(-c) &= \frac{-\mathcal{D}^2 - d^2 - \sqrt{[\mathcal{D}^2 + d^2]^2 - 4d^2\mathcal{D}^2}}{2d\mathcal{D}} = \frac{-\mathcal{D}^2 - d^2 - \sqrt{[\mathcal{D}^2 - d^2]^2}}{2d\mathcal{D}} \\ &= \frac{-\mathcal{D}^2 - d^2 - |\mathcal{D}^2 - d^2|}{2d\mathcal{D}}. \end{aligned}$$

Suppose that the L-system $\Theta_{\mu,h}$ with a given coefficient of dissipation $\mathcal{D} = \operatorname{Im} h \neq \operatorname{Im} m_\infty(i)$ has the maximum finite c-entropy $\mathcal{S}_{max}$. As we already mentioned in the proof of Theorem 21, the maximum finite c-entropy is achieved when the modulus of the von Neumann parameter κ is at its minimum. Since $a > 1$, the corresponding κ is related to a via (3.7). Consider κ as a function of a single variable x. Then

$$\kappa(x) = \frac{a(x) - 1}{1 + a(x)}$$

is increasing function of a and hence $\kappa(x)$ takes its minimum whenever $a(x)$ assumes its minimum. But as we have just shown above $a(x)$ has its absolute minimum at

$x = -c$. Therefore, the L-system $\Theta_{\mu,h}$ has the maximum c-entropy $\mathcal{S}_{max}$ when $x = -c = -\text{Re}\, m_\infty(i)$. The corresponding values of system parameters h and μ are found once we recall that $x = \text{Re}\, h$ and $\mathcal{D} = \text{Im}\, h$ and then substitute $x = -c = -\text{Re}\, m_\infty(i)$ in (8.2) considering both possible cases for a in (8.16). This proves necessity.

Conversely, let the L-system $\Theta_{\mu,h}$ has the parameters h and μ defined by (8.14). Then as it follows from the first part of (8.14) we have $\text{Re}\, h = x = -\text{Re}\, m_\infty(i) = -c$. As we have shown in the first part of the proof, this value of $x = -c$ minimizes $a(x)$, minimizes $\kappa(x)$, and hence maximizes c-entropy of our L-system. Thus, $\Theta_{\mu,h}$ has the maximum finite c-entropy $\mathcal{S}_{max}$ if (8.14) holds.

In order to prove (8.10) we observe that (8.13) is still true for this case when one substitutes the values of $a(-c)$ from (8.16) into (3.7). Hence (8.13) again determines the absolute minimum of κ for all real x. It follows from (7.2) that the minimum of κ corresponds to the maximum of c-entropy which confirms (8.10). □

We point out that both Theorems 21 and 22 describe uniquely the L-systems with maximum c-entropy for the corresponding classes of impedance functions. Also, an attentive reader notices that in Theorems 21 and 22 the hypothesis that $\mathcal{D} = \text{Im}\, h \neq -\text{Im}\, m_\infty(i)$ holds and might wonder what happens if $h = -m_\infty(i)$ for a given L-system. The following remark addresses that question.

Remark 23 An L-system $\Theta_{\mu,h}$ of the form (4.8) with the Schrödinger operator T_h and parameter $\mu \in \mathbb{R} \cup \{\infty\}$ has infinite c-entropy, $\mathcal{S} = \infty$ if and only if $h = -m_\infty(i)$. Indeed, one might also notice that if $h = -m_\infty(i)$, then $\mathcal{D} = \text{Im}\, h = -\text{Im}\, m_\infty(i)$ and hence formulas (8.12) and (8.16) imply that $a(-c) = 1$ and $\kappa(-c) = 0$. The assertion then is a corollary of Theorem 8.

9 c-Entropy in Sectorial Cases and the Krein–von Neumann Extension

Throughout this section we deal with L-systems whose main operators are either extremal accretive or β-sectorial.

9.1 Dual c-Entropy Problems in the Extremal Case

The following theorem gives the solution to the first dual c-entropy problem posed in Sect. 7.

Theorem 24 *Assume that an L-system $\Theta_{\mu,h}$ of the form (4.8) with the main operator T_h has c-entropy $\mathcal{S}$. Then, under the constraint that T_h is extremal, the L-system $\Theta_{\mu,h}$ has the minimum coefficient of dissipation*

$$\mathcal{D}_{min} = -\text{Im}\, m_\infty(i) \coth \mathcal{S} - \sqrt{B^2 \operatorname{csch}^2 \mathcal{S} - (A - m)^2}, \tag{9.1}$$

where $A = Re\, m_\infty(i)$, $B = -\text{Im}\, m_\infty(i)$, and $m = m_\infty(-0)$.

In this case, the L-system $\Theta_{\mu,h}$ is determined by the parameters

$$h = -m_\infty(-0) + i\mathcal{D}_{min} \quad and \quad arbitrary \quad \mu \in \mathbb{R} \cup \{\infty\}. \tag{9.2}$$

Proof Suppose that the Schrödinger L-system $\Theta_{\mu,h}$ has a given fixed c-entropy $\mathcal{S}$. Then, as we have shown in Sect. 5, the modulus of von Neumann parameter κ is related to the coefficient of dissipation $\mathcal{D} = \operatorname{Im} h$ via (5.7)

$$\kappa^2 = \frac{\mathcal{D}^2 - 2B\mathcal{D} + D}{\mathcal{D}^2 + 2B\mathcal{D} + D}, \tag{9.3}$$

where $B = -\operatorname{Im} m_\infty(i)$, $D = C + B^2$, and $C = (\operatorname{Re} m_\infty(i) - m_\infty(-0))^2$. By Theorem 12, $\kappa_0 \le \kappa < 1$, where κ_0 is given by (5.9). Since $\kappa = e^{-\mathcal{S}}$, by (9.3), we have

$$e^{-2\mathcal{S}} = \frac{\mathcal{D}^2 - 2B\mathcal{D} + D}{\mathcal{D}^2 + 2B\mathcal{D} + D}$$

and hence

$$(1 - e^{-2\mathcal{S}})\mathcal{D}^2 - 2B(1 + e^{-2\mathcal{S}})\mathcal{D} + (1 - e^{-2\mathcal{S}})D = 0,$$

or

$$\mathcal{D}^2 - 2B(\coth \mathcal{S})\mathcal{D} + D = 0. \tag{9.4}$$

This equation (see the proof of Theorem 14) has either two or one real roots (also see Fig. 1 for convenience) given by

$$\mathcal{D} = B \coth \mathcal{S} \pm \sqrt{B^2 \coth^2 \mathcal{S} - D}, \tag{9.5}$$

where B and D are defined above. The case of a unique solution occurs when $e^{-\mathcal{S}} = \kappa = \kappa_0$. As we have shown in Sect. 5, in this case

$$\mathcal{D} = \sqrt{D} = \sqrt{(\operatorname{Im} m_\infty(i) - m_\infty(-0))^2 + (\operatorname{Im} m_\infty(i))^2}.$$

If $\kappa_0 < \kappa < 1$, then the minimal coefficient of dissipation is the smallest of two roots of equation (9.5). It is easy to see that

$$\mathcal{D}_{min} = B \coth \mathcal{S} - \sqrt{B^2 \coth^2 \mathcal{S} - D}$$

in this case. Substituting the values for B and D into the above formula and taking into account that $\operatorname{csch}^2 \mathcal{S} = \coth^2 \mathcal{S} - 1$ we obtain (9.1). □

Now we address the second dual c-entropy problem.

Theorem 25 *Assume that an L-system $\Theta_{\mu,h}$ is of the form (4.8) with a main operator T_h. Assume in addition that Re $m_\infty(i) \neq m_\infty(-0)$. Then, under the constraint that*

T_h is extremal, the L-system $\Theta_{\mu,h}$ has the maximum finite c-entropy $\mathcal{S}_{max}$ if and only if it is determined by the parameters

$$h = -m + i\sqrt{(A-m)^2 + B^2} \quad and \quad (arbitrary) \quad \mu \in \mathbb{R} \cup \{\infty\}, \tag{9.6}$$

where $A = Re\, m_\infty(i)$, $B = -\operatorname{Im} m_\infty(i)$, and $m = m_\infty(-0)$.

In this case,

$$\mathcal{S}_{max} = -\frac{1}{2}\ln\left(\frac{\sqrt{A^2 + B^2 - 2mA + m^2} - B}{\sqrt{A^2 + B^2 - 2mA + m^2} + B}\right). \tag{9.7}$$

Proof It follows from the development in Sect. 5 that there are many L-system of the form (4.8) with an extremal main operators T_h only different by the coefficient of dissipation (see also Fig. 1). Naturally, such an L-system has the maximum c-entropy $\mathcal{S}_{max}$ if the modulus of von Neumann parameter κ of T_h is at minimum, i.e., $\kappa = \kappa_0$ (see formula (5.9)). Note that $\kappa_0 \neq 0$ or otherwise $A = m$ and our L-system $\Theta_{\mu,h}$ has infinite entropy according to Theorem 8. We know that $\kappa = \kappa_0$ if and only if

$$\operatorname{Im} h = \sqrt{(\operatorname{Re} m_\infty(i) - m_\infty(-0))^2 + (\operatorname{Im} m_\infty(i))^2} = \sqrt{(A-m)^2 + B^2}. \tag{9.8}$$

In this case (see (7.2)) $\mathcal{S}_{max} = -\ln(\kappa_0)$ proving (9.7). □

Remark 26 If $\operatorname{Re} m_\infty(i) = m_\infty(-0)$, then $h = -m_\infty(i)$ and our L-system $\Theta_{\mu,h}$ has infinite entropy according to Theorem 8.

9.2 Dual c-Entropy Problems in the Sectorial Case

We use similar analysis to treat the sectorial case. We remind the reader that by saying that an accretive operator is β-sectorial, we mean that $\beta \in (0, \pi/2)$ is its exact angle of sectoriality unless otherwise is specified.

The following theorem is analogues to Theorem 24 for the sectorial case.

Theorem 27 *Assume that an L-system $\Theta_{\mu,h}$ of the form (4.8) with the main operator T_h has c-entropy $\mathcal{S}$. Then, under the constraint that T_h is β-sectorial, the L-system $\Theta_{\mu,h}$ has the minimum coefficient of dissipation*

$$\mathcal{D}_{min} = \sin^2\beta\left[B\coth\mathcal{S} - E\cot\beta - \sqrt{(E\cot\beta - B\coth\mathcal{S})^2 - D}\right], \tag{9.9}$$

where $A = Re\, m_\infty(i)$, $B = -\operatorname{Im} m_\infty(i)$, $D = E^2 + B^2$, $E = A - m$, and $m = m_\infty(-0)$.

In this case, the L-system $\Theta_{\mu,h}$ is determined by the parameters

$$h = (\cot\beta)\mathcal{D}_{min} - m + i\mathcal{D}_{min} \quad and \quad (arbitrary) \quad \mu \in \mathbb{R} \cup \{\infty\}. \tag{9.10}$$

Proof Let $\Theta_{\mu,h}$ have a β-sectorial main operator T_h whose von Neumann's parameter is κ. Then κ is related to the coefficient of dissipation $\mathcal{D} = \operatorname{Im} h$ via (6.2), that is

$$\kappa^2 = \frac{\mathcal{D}^2 + 2\sin^2\beta((A-m)\cot\beta - B)\mathcal{D} + D\sin^2\beta}{\mathcal{D}^2 + 2\sin^2\beta((A-m)\cot\beta + B)\mathcal{D} + D\sin^2\beta}. \tag{9.11}$$

According to Theorem 15, the parameter κ in this case belongs to the interval $0 < \kappa_0 \le \kappa < 1$, where κ_0 is given by (6.3), (6.4). Note that $\kappa_0 \neq 0$ since otherwise this will contradict Lemma 2.

In order to find the minimal coefficient of dissipation we recall that $\kappa = e^{-\mathcal{S}}$, use (9.11), and solve

$$e^{-2\mathcal{S}} = \frac{\mathcal{D}^2 + 2\sin^2\beta((A-m)\cot\beta - B)\mathcal{D} + D\sin^2\beta}{\mathcal{D}^2 + 2\sin^2\beta((A-m)\cot\beta + B)\mathcal{D} + D\sin^2\beta}$$

for $\mathcal{D}$ to pick the minimal root. The above equation simplifies to the quadratic one of the form

$$\begin{aligned}(1-e^{-2\mathcal{S}})\mathcal{D}^2 + 2B\sin^2\beta\left[(1-e^{-2\mathcal{S}})(A-m) - \left(1+e^{-2\mathcal{S}}\right)B\right]\mathcal{D}\\ + (1-e^{-2\mathcal{S}})D\sin^2\beta = 0,\end{aligned}$$

or, with $E = A - m$,

$$\mathcal{D}^2 + 2B\sin^2\beta\,[E\cot\beta - (\coth\mathcal{S})\,B]\,\mathcal{D} + D\sin^2\beta = 0. \tag{9.12}$$

Equation (9.12) has either two or one real solutions (see also Fig. 2) given by

$$\mathcal{D} = (\sin^2\beta)\left[B\coth\mathcal{S} - E\cot\beta \pm \sqrt{(E\cot\beta - B\coth\mathcal{S})^2 - D}\right]. \tag{9.13}$$

The case of a unique solution occurs when $e^{-\mathcal{S}} = \kappa = \kappa_0$, where κ_0 is given by (6.3), (6.4). As we have shown in Sect. 6, in this case

$$\mathcal{D} = (\sin\beta)\sqrt{D} = (\sin\beta)\sqrt{|m_\infty(i)|^2 - 2m_\infty(-0)\mathrm{Re}\ m_\infty(i) + m_\infty^2(-0)}.$$

If $\kappa_0 < \kappa < 1$, then the minimal coefficient of dissipation is the smallest of the two roots of (9.13). It is easy to see that in this case

$$\mathcal{D}_{min} = (\sin\beta)^2\left[B\coth\mathcal{S} - E\cot\beta - \sqrt{(E\cot\beta - B\coth\mathcal{S})^2 - D}\right]. \tag{9.14}$$

Formula (9.10) follows from the relation (6.1) connecting the real and imaginary part of the parameter h that describes a β-sectorial operator T_h. □

Now we address the second dual problem of maximizing c-entropy for the class of Schrödinger L-systems with sectorial main operators.

The following theorem is similar to Theorem 25.

Theorem 28 *Assume that an L-system $\Theta_{\mu,h}$ is of the form (4.8) with the main operator T_h. Then, under the constraint that T_h is β-sectorial, the L-system $\Theta_{\mu,h}$ has the maximum finite c-entropy $\mathcal{S}_{max}$ if and only if it is determined by the parameters*

$$h = (\cos\beta)\sqrt{D} - m + i(\sin\beta)\sqrt{D} \quad and \quad (arbitrary) \quad \mu \in \mathbb{R} \cup \{\infty\}, \tag{9.15}$$

where $A = Re\, m_\infty(i)$, $B = -\operatorname{Im} m_\infty(i)$, $D = (A-m)^2 + B^2$, and $m = m_\infty(-0)$.

In this case,

$$\mathcal{S}_{max} = -\frac{1}{2}\ln\left(\frac{\sqrt{D} + (A-m)\cos\beta - B\sin\beta}{\sqrt{D} + (A-m)\cos\beta + B\sin\beta}\right). \tag{9.16}$$

Proof It follows from the development in Sect. 6 and formulas (6.3) and (6.4) that an L-system of the form (4.8) with a β-sectorial main operator T_h has the maximum c-entropy $\mathcal{S}_{max}$ if the modulus of von Neumann parameter κ of T_h is at minimum (see also Fig. 2). That happens if and only if $\kappa = \kappa_0$ (see (6.3) and (6.4)) and hence

$$\operatorname{Im} h = \sin\beta \cdot \sqrt{D} = \sin\beta \cdot \sqrt{(\operatorname{Re} m_\infty(i) - m_\infty(-0))^2 + (\operatorname{Im} m_\infty(i))^2}. \tag{9.17}$$

Note that $\kappa_0 \neq 0$ or otherwise this will violate Lemma 2. Moreover, in this case (see (7.2)) $\mathcal{S}_{max} = -\ln(\kappa_0)$ proving (9.16). The expression in the right hand side of (9.16) is well defined as the fraction inside the logarithm represents κ_0^2, the positive quantity. □

9.3 Accretive T_h Case

In this subsection we are going to look at the situation when the main operator T_h of the Schrödinger L-system under consideration is just accretive. Recall (see Sect. 2 for the definition) that the operator T_h is accretive if $\operatorname{Re}(T_h y, y) \geq 0$ for all $y \in \operatorname{Dom}(T_h)$. Clearly, the set of all such accretive operators consists of the class of β-sectorial (for some $\beta \in (0, \pi/2)$) operators plus the class of extremal operators.

The first question we address is which L-systems $\Theta_{\mu,h}$ with accretive main operators T_h have the maximal finite c-entropy.

Theorem 29 *Assume that an L-system $\Theta_{\mu,h}$ is of the form (4.8) with the main operator T_h. Then, under the constraint that T_h is accretive, the L-system $\Theta_{\mu,h}$ with extremal accretive main operator achieves the maximum finite c-entropy $\mathcal{S}_{max}$.*

In this case, $\mathcal{S}_{max}$ is given by (9.7) and $\Theta_{\mu,h}$ is determined by the parameters

$$h = -m + i\sqrt{(A-m)^2 + B^2} \quad and \quad (arbitrary)\ \mu \in \mathbb{R} \cup \{\infty\}, \tag{9.18}$$

where $A = Re\, m_\infty(i)$, $B = -\operatorname{Im} m_\infty(i)$, *and* $m = m_\infty(-0)$.

Proof We have already mentioned above that since our main operator T_h is accretive, then it is either β-sectorial (for some $\beta \in (0, \pi/2)$) or extremal accretive. In the first case the L-system $\Theta_{\mu,h}$ with β-sectorial main operator has the maximum entropy $\mathcal{S}^{sec}_{max}$ given by (9.16). In the second case the L-system $\Theta_{\mu,h}$ with the extremal main operator has the maximum entropy $\mathcal{S}^{ext}_{max}$ given by (9.7).

In order to prove the theorem we need to confirm that the latter is larger, that is

$$\begin{aligned}\mathcal{S}^{sec}_{max} &= -\frac{1}{2}\ln\left(\frac{\sqrt{D}+(A-m)\cos\beta - B\sin\beta}{\sqrt{D}+(A-m)\cos\beta + B\sin\beta}\right) < \mathcal{S}^{ext}_{max}\\ &= -\frac{1}{2}\ln\left(\frac{\sqrt{D}-B}{\sqrt{D}+B}\right),\end{aligned}$$

for any $\beta \in (0, \pi/2)$. Here $D = (A-m)^2 + B^2$. This inequality is clearly equivalent to the simpler one

$$\frac{\sqrt{D}+(A-m)\cos\beta - B\sin\beta}{\sqrt{D}+(A-m)\cos\beta + B\sin\beta} > \frac{\sqrt{D}-B}{\sqrt{D}+B}, \quad \forall \beta \in (0, \pi/2). \tag{9.19}$$

We bring attention of the reader to the parameters A and m in the inequality (9.19). As we have shown in Lemma 17, $A \geq m$. If we assume that $A = m$, then it follows from Lemma 13 that the minimal von Neumann's parameter κ_0 associated with the extremal operator T_h is such that $\kappa_0 = 0$ and hence the maximum c-entropy of L-system with such extremal T_h is infinite, i.e., $\mathcal{S}^{ext}_{max} = \infty$. In this theorem we are only interested in the case of finite maximum c-entropy and thus the case when $A = m$ should not be considered. Thus, we can assume without loss of generality that $A > m$.

Consider the function

$$f(\beta) = \frac{\sqrt{D}+(A-m)\cos\beta - B\sin\beta}{\sqrt{D}+(A-m)\cos\beta + B\sin\beta} \quad \text{on} \quad \beta \in (0, \pi/2). \tag{9.20}$$

Since $A > m$, we have that the derivative

$$f'(\beta) = -\frac{2B\left(A-m+\sqrt{D}\cdot\cos\beta\right)}{\left(\sqrt{D}+(A-m)\cos\beta + B\sin\beta\right)^2} < 0 \tag{9.21}$$

for any $\beta \in (0, \pi/2)$. Hence the function $f(\beta)$ in (9.20) decreases on $\beta \in (0, \pi/2)$ and

$$f\left(\frac{\pi}{2}-\right) = \frac{\sqrt{D}-B}{\sqrt{D}+B}.$$

Note that under our current assumption $f(\pi/2-) \neq 0$ or otherwise $A = m$ (see Lemma 13).

Therefore, inequality (9.19) holds for $A > m$. Consequently, the Schrödinger L-system $\Theta_{\mu,h}$ with an accretive main operator T_h achieves the maximum finite c-entropy whenever T_h is extremal. In this case, according to Theorem 25, $\mathcal{S}_{max}$ is determined by (9.7) while h is given by (9.18) and μ is arbitrary in $\mathbb{R} \cup \{\infty\}$. □

Remark 30 Note that if $\Theta_{\mu,h}$ is the Schrödinger L-system referred to in Theorem 29, then the modulus of von Neumann's parameter of the corresponding main operator T_h equals κ_0 given by (5.9). Moreover, if under the assumptions of Theorem 29, $\Theta_{\mu_1,h}$ and $\Theta_{\mu_2,h}$ are such L-systems that the corresponding impedance functions $V_{\Theta_{\mu_1,h}}(z)$ and $V_{\Theta_{\mu_2,h}}(z)$ belong to he generalized Donoghue classes $\mathfrak{M}_{\kappa_0}$ or $\mathfrak{M}_{\kappa_0}^{-1}$ respectively, then

$$\mu_1 = \frac{(A - mBF)(BF - 1) + (m^2F + DF - B - mAF)(A - m)F}{(BF - 1)^2 + (A - m)^2F^2} \tag{9.22}$$

and

$$\mu_2 = -\frac{\mu_1 m + m^2 + (A - m)^2 + B^2}{\mu_1 + m}. \tag{9.23}$$

Here

$$F = \frac{\sqrt{D} - \sqrt{C}}{B\sqrt{D}}$$

and all the other letters are defined above.

Indeed, let the L-systems $\Theta_{\mu_1,h}$ and $\Theta_{\mu_2,h}$ be such that

$$V_{\Theta_{\mu_1,h}}(i) = ai \quad \text{and} \quad V_{\Theta_{\mu_2,h}}(i) = \left(\frac{1}{a}\right) i, \tag{9.24}$$

where

$$a = \frac{1 - \kappa_0}{1 + \kappa_0} = \frac{1 - \sqrt{\frac{\sqrt{D}-B}{\sqrt{D}+B}}}{1 + \sqrt{\frac{\sqrt{D}-B}{\sqrt{D}+B}}} = \frac{\sqrt{D} - \sqrt{D - B^2}}{B} = \frac{\sqrt{D} - \sqrt{C}}{B}.$$

Then (9.24) will guarantee that for the corresponding impedance functions we have $V_{\Theta_{\mu_1,h}}(z) \in \mathfrak{M}_{\kappa_0}$ and $V_{\Theta_{\mu_2,h}}(z) \in \mathfrak{M}_{\kappa_0}^{-1}$. By (4.10)

$$V_{\Theta_{\mu,h}}(i) = \frac{(A - Bi + \mu)\sqrt{D}}{(\mu + m)(A - Bi) - \mu m - m^2 - D}.$$

In view of (9.24) we have the equation

$$\frac{(A-Bi+\mu)\sqrt{D}}{(\mu+m)(A-Bi)-\mu m-m^2-D}=\frac{\sqrt{D}-\sqrt{C}}{B}i,$$

which we are going to solve for μ_1. To simplify the calculation process we set $F=a/\sqrt{D}=\frac{\sqrt{D}-\sqrt{C}}{B\sqrt{D}}$ and hence

$$\begin{aligned}\mu_1&=\frac{A-mBF+(m^2F+DF-B-mAF)i}{BF-1+(A-m)Fi}\\&=\frac{(A-mBF)(BF-1)+(m^2F+DF-B-mAF)(A-m)F}{(BF-1)^2+(A-m)^2F^2}\\&\quad+i\,\frac{(m^2F+DF-B-mAF)(BF-1)-(A-mBF)(A-m)F}{(BF-1)^2+(A-m)^2F^2}.\end{aligned}\tag{9.25}$$

Substituting $D=(A-m)^2+B^2$ and $F=\frac{\sqrt{D}-\sqrt{C}}{B\sqrt{D}}$ we confirm that the imaginary part of μ_1 in (9.25) is zero. Thus,

$$\mu_1=\frac{(A-mBF)(BF-1)+(m^2F+DF-B-mAF)(A-m)F}{(BF-1)^2+(A-m)^2F^2}.\tag{9.26}$$

To find μ_2 we use (7.12) and the reasoning in the proof of Theorem 20 (see also [2]) and get

$$\mu_2=\frac{\mu_1\mathrm{Re}\,h-|h|^2}{\mu_1-\mathrm{Re}\,h}=-\frac{\mu_1 m+m^2+(A-m)^2+B^2}{\mu_1+m}.\tag{9.27}$$

Thus, the Schrödinger L-systems $\Theta_{\mu_1,h}$ and $\Theta_{\mu_2,h}$ with the desired properties are uniquely constructed using the values of h from (9.18), μ_1 from (9.22), and μ_2 from (9.23).

9.4 Maximal c-Entropy in Accretive Case

In this subsection we consider Schrödinger L-systems with maximal finite c-entropy and accretive state-space operator $\mathbb{A}$. The following theorem describes an L-system with β-sectorial main and state-space operators and maximal c-entropy.

Theorem 31 *An L-system $\Theta_{\mu,h}$ of the form (4.8) with a β-sectorial ($\beta\in(0,\pi/2)$) Schrödinger main operator T_h and the maximum finite c-entropy $\mathcal{S}_{max}$ determined by the formula (9.16) has a β-sectorial (with the same angle of sectoriality) state-space operator $\mathbb{A}_{\mu,h}$ if and only if $\mu=\infty$ in (4.3).*

Proof It was shown in [2, Theorem 10.6.4] (see also [5, 8]) that if $\mathbb{A}_{\mu,h}$ is a $(*)$-extension of an β-sectorial operator T_h with the exact angle of sectoriality $\beta\in$

$(0, \pi/2)$, then $\mathbb{A}_{\mu,h}$ is an β-sectorial $(*)$-extension of T_h (with the same angle of sectoriality) if and only if $\mu = +\infty$. The rest follows from Theorem 28. □

Theorem 32 *An L-system $\Theta_{\mu,h}$ of the form (4.8) with a β-sectorial ($\beta \in (0, \pi/2)$) Schrödinger main operator T_h and the maximum finite c-entropy $\mathcal{S}_{max}$ determined by the formula (9.16) has an extremal state-space operator $\mathbb{A}_{\mu,h}$ if and only if*

$$\mu = (2\csc 2\beta)\sqrt{(Re\, m_\infty(i) - m_\infty(-0))^2 + (\operatorname{Im} m_\infty(i))^2} - m_\infty(-0). \quad (9.28)$$

Proof It follows from [2, Theorem 10.6.5] (see also [5, 8]) that if $\mathbb{A}_{\mu,h}$ is a $(*)$-extension of an β-sectorial operator T_h with the exact angle of sectoriality $\beta \in (0, \pi/2)$, then $\mathbb{A}_{\mu,h}$ is accretive but not β-sectorial for any $\beta \in (0, \pi/2)$ $(*)$-extension of T_h if and only if in (4.3)

$$\mu = \frac{(\operatorname{Im} h)^2}{m_\infty(-0) + \operatorname{Re} h} + \operatorname{Re} h. \quad (9.29)$$

Taking into account that in our case $\operatorname{Im} h$ and $\operatorname{Re} h$ are related by (6.1), we substitute their values into (9.29) to get

$$\begin{aligned} \mu &= \frac{(\operatorname{Im} h)^2}{m + \operatorname{Re} h} + \operatorname{Re} h = \frac{(\operatorname{Im} h)^2}{m + (\cot\beta)\operatorname{Im} h - m} + (\cot\beta)\operatorname{Im} h - m \\ &= (\tan\beta)\operatorname{Im} h + (\cot\beta)\operatorname{Im} h - m = (\tan\beta + \cot\beta)\operatorname{Im} h - m \\ &= (2\csc 2\beta)\operatorname{Im} h - m, \end{aligned}$$

where $m = m_\infty(-0)$. We know from Theorem 28 that if the L-system $\Theta_{\mu,h}$ has the maximum c-entropy, then $\operatorname{Im} h$ is given by (9.17). Substituting the value of $\operatorname{Im} h$ into the above we obtain (9.28). □

Now we focus on the case when the main operator of a Schrödinger L-systen is extremal.

Theorem 33 *An L-system $\Theta_{\mu,h}$ of the form (4.8) with an extremal Schrödinger main operator T_h and the maximum finite c-entropy $\mathcal{S}_{max}$ determined by the formula (9.7) has an accretive state-space operator $\mathbb{A}_{\mu,h}$ if and only if $\mu = \infty$.*

Proof As it was shown in [2, Theorem 10.6.4] (see also [5, 8]), a state-space operator $\mathbb{A}_{\mu,h}$ preserves the exact angle of sectoriality of T_h if and only if $\mu = \infty$. Here our operator T_h is extremal and thus can only admit one extremal $(*)$-extension $\mathbb{A}_{\mu,h}$ with $\mu = \infty$. □

At this point again we are going to look at the combined class of β-sectorial and extremal accretive operators to see what case provides us with an L-system with maximal finite c-entropy.

Theorem 34 *An L-system $\Theta_{\mu,h}$ of the form (4.8) with an accretive Schrödinger main operator T_h and accretive state-space operator $\mathbb{A}_{\mu,h}$ achieves the maximum finite c-entropy $\mathcal{S}_{max}$ determined by (9.7) when T_h is extremal.*

In this case h is given by (9.18), $\mu = \infty$, and the quasi-kernel of Re $\mathbb{A}_{\mu,h}$ is the Krein-von Neumann extension defined by the formula

$$\begin{cases} A_K y = -y'' + q(x)y, \\ y'(\ell) + m_\infty(-0)y(\ell) = 0. \end{cases} \tag{9.30}$$

Proof We know that according to Theorem 29 among all the L-system $\Theta_{\mu,h}$ of the form (4.8) with an accretive Schrödinger main operator T_h and any $\mu \in \mathbb{R} \cup \{\infty\}$, the L-system that has the maximum finite c-entropy is the one with the extremal main operator T_h. On the other hand, since we are given an extra condition that the state-space operator $\mathbb{A}_{\mu,h}$ is also accretive, then Theorem 33 provides us with $\mu = \infty$ for this case. Consequently, $\mathbb{A}_{\mu,h}$ is given by (7.13) that is

$$\begin{aligned} \mathbb{A}_{\mu,h} &= \mathbb{A}_{\infty,h}\, y = -y'' + q(x)y - [y'(\ell) - hy(\ell)]\,\delta(x-\ell), \\ \mathbb{A}^*_{\mu,h} &= \mathbb{A}^*_{\infty,h}\, y = -y'' + q(x)y - [y'(\ell) - \overline{h}\, y(\ell)]\,\delta(x-\ell). \end{aligned} \tag{9.31}$$

Using (9.31) one easily gets

$$\text{Re } \mathbb{A}_{\infty,h} = -y'' + q(x)y + \left[y'(\ell) + m_\infty(-0)y(\ell)\right]\delta(x-\ell). \tag{9.32}$$

Clearly, (9.32) implies that Re $\mathbb{A}_{\infty,h} \supset A_K$ and A_K given by (9.30) is the quasi-kernel for Re $\mathbb{A}_{\infty,h}$. □

Remark 35 Note that if an L-system $\Theta_{\mu,h}$ of the form (4.8) has an accretive state-space operator, then its impedance function $V_{\Theta_{\mu,h}}(z)$ cannot belong to any of the generalized Donoghue classes.

Indeed, we know from [2, Theorem 9.8.2] that if an L-system has an accretive state-space operator, then its impedance function is a Stieltjes function and thus has the integral representation

$$V_{\Theta_{\mu,h}}(z) = \gamma + \int_0^\infty \frac{d\sigma(t)}{t-z}, \tag{9.33}$$

where $\gamma \geq 0$ and $\sigma(t)$ is a non-decreasing on $[0, +\infty)$ function such that $\int_0^\infty \frac{dG(t)}{1+t} < \infty$. On the other hand, if we assume that a Stieltjes function $V_{\Theta_{\mu,h}}(z)$ belonged to a generalized Donoghue class, then it would have admited integral representation of the form (3.2), that is

$$V_{\Theta_{\mu,h}}(z) = \int_0^{+\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\sigma(t). \tag{9.34}$$

However, calculating the real part of $V_{\Theta_{\mu,h}}(i)$ using both (9.33) and (9.34) and then setting the results equal leads to

$$\mathrm{Re}\, V_{\Theta_{\mu,h}}(i) = 0 = \gamma + \int\limits_0^\infty \frac{t\,d\sigma(t)}{t^2+1},$$

which is a contradiction since $\gamma \geq 0$ and $\int_0^\infty \frac{t\,d\sigma(t)}{t^2+1} > 0$ (see [2]). Therefore we have shown that a Stieltjes function $V_{\Theta_{\mu,h}}(z)$ can be written as

$$V_{\Theta_{\mu,h}}(z) = Q + \int_0^{+\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\sigma(t),$$

with

$$Q = \gamma + \int\limits_0^\infty \frac{t\,d\sigma(t)}{t^2+1} > 0. \tag{9.35}$$

Consequently, none of the Stieltjes impedance functions $V_{\Theta_{\mu,h}}(z)$ can belong to any of the generalized Donoghue classes.

10 Table of Dual c-Entropy Problems Solutions

In this subsection we summarize the results on dual c-entropy optimization problems and present them in the tabular form (see Table 1).

In the very left column of the table we list the properties of the Schrödinger L-systems under consideration. The second and the third columns of the table indicate the theorems that contain solutions to the first and the second dual c-entropy problems, respectively. The last column of Table 1 addresses the uniqueness of the Schrödinger L-system solving either of the dual c-entropy problems. We have already noted in Sects. 7 and 8 that the Schrödinger L-systems $\Theta_{\mu,h}$ referred to in Theorems 19–22 are unique with the exact values of the defining parameters μ and h provided. On the other hand, when the dual c-entropy problems are being solved for the classes of Schrödinger L-systems with extremal or β-sectorial main operators T_h the answer is

Table 1 Solutions to dual c-entropy problems

L-system property	First dual problem	Second dual problem	L-system uniqueness
$V_\Theta(z) \in \mathfrak{M}_\kappa$	Theorem 20	Theorem 21	Unique
$V_\Theta(z) \in \mathfrak{M}_\kappa^{-1}$	Theorem 19	Theorem 22	Unique
T_h is extremal	Theorem 24	Theorem 25	Not unique
T_h is β-sectorial	Theorem 27	Theorem 28	Not unique

not unique. Namely, Theorems 24–28 describe the entire family of solutions. These solutions $\Theta_{\mu,h}$ are parameterized by a particular value of h and an arbitrary parameter $\mu \in \mathbb{R} \cup \{\infty\}$. Note that the case when the impedance function $V_\Theta(z)$ belongs to the Donoghue class $\mathfrak{M}$ deals with infinite c-entropy (see Theorem 8) and therefore is not presented in the table.

In the next section we will illustrate our findings by two examples.

11 Examples

We conclude this paper with a couple of simple illustrations. Consider the differential expression

$$l_\nu = -\frac{d^2}{dx^2} + \frac{\nu^2 - 1/4}{x^2}, \quad x \in [1, \infty)$$

of order $\nu \geq \frac{1}{2}$ in the Hilbert space $\mathcal{H} = L^2[1, \infty)$. The minimal symmetric operator

$$\begin{cases} \dot{A}\, y = -y'' + \frac{\nu^2 - 1/4}{x^2} y \\ y(1) = y'(1) = 0 \end{cases} \tag{11.1}$$

generated by this expression and boundary conditions has deficiency indices (1, 1) and is obviously nonnegative. Consider also the operator

$$\begin{cases} T_h\, y = -y'' + \frac{\nu^2 - 1/4}{x^2} y \\ y'(1) = hy(1). \end{cases} \tag{11.2}$$

Example 1

Let $\nu = 1/2$. It is known [2] that in this case

$$m_{\infty,\frac{1}{2}}(z) = -i\sqrt{z} \quad \text{and hence} \quad m_{\infty,\frac{1}{2}}(i) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i.$$

The minimal symmetric operator then becomes

$$\begin{cases} \dot{A}\, y = -y'' \\ y(1) = y'(1) = 0. \end{cases}$$

Note that in this case

$$\operatorname{Re} m_{\infty,\frac{1}{2}}(i) = \frac{1}{\sqrt{2}} > m_{\infty,\frac{1}{2}}(0) = 0,$$

which taking into account that the symmetric operator $\dot{A}$ is nonnegative, illustrates the result of Lemma 17.

To construct a family of L-systems with von Neumann's parameter $\kappa = 0$ for this case we set

$$h = -m_{\infty,\frac{1}{2}}(i) = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i.$$

Then the main operator is given by

$$\begin{cases} T_{h_0}\, y = -y'' \\ y'(1) = \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) y(1) \end{cases} \tag{11.3}$$

and it will be shared by all the family of L-systems with $\kappa = 0$ and

$$h_0 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i. \tag{11.4}$$

Clearly, any L-system Θ_{μ,h_0} has infinite c-entropy for all $\mu \in \mathbb{R} \cup \{\infty\}$. The quasi-kernel of the real part of the state-space operator of this family of L-systems is determined by (4.6) as follows

$$\begin{cases} \hat{A}_\mu\, y = -y'' \\ y'(1) = -\frac{\mu+\sqrt{2}}{\sqrt{2}\mu+1} y(1). \end{cases} \tag{11.5}$$

To construct an L-system with an accretive state-space operator, we take $\mu = \infty$ to have

$$\begin{cases} \hat{A}_\infty\, y = -y'' \\ y'(1) = -\frac{1}{\sqrt{2}} y(1). \end{cases}$$

The state-space operator of the L-system Θ_{∞,h_0} with $\kappa = 0$, h_0 defined by (11.4), and $\mu = \infty$ is (see (4.3))

$$\begin{aligned} \mathbb{A}_{\infty,h_0}\, y &= -y'' - \frac{1}{\sqrt{2}}[\sqrt{2}y'(1) + (1-i)\, y(1)]\, \delta(x-1), \\ \mathbb{A}^*_{\infty,h_0}\, y &= -y'' - \frac{1}{\sqrt{2}}[\sqrt{2}y'(1) + (1+i)\, y(1)]\, \delta(x-1). \end{aligned} \tag{11.6}$$

Also the channel operator $K_{\infty,h_0}\, c = c g_{\infty,h_0}$, $(c \in \mathbb{C})$, where (see (4.4))

$$g_{\infty,h_0} = 2^{-\frac{1}{4}} \delta(x-1).$$

Then Θ_{∞,h_0} has the form

$$\Theta_{\infty,h_0} = \begin{pmatrix} \mathbb{A}_{\infty,h_0} & K_{\infty,h_0} & 1 \\ \mathcal{H}_+ \subset L_2[1,+\infty) \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix}, \tag{11.7}$$

where all the components are described above. Using formulas (4.9) and (4.10) we obtain

$$W_{\Theta_{\infty,h_0}}(z)=\frac{m_{\infty,\frac{1}{2}}(z)+\bar{h}}{m_{\infty,\frac{1}{2}}(z)+h}=\frac{i\sqrt{2z}+1+i}{i\sqrt{2z}+1-i} \tag{11.8}$$

and

$$V_{\Theta_{\infty,h_0}}(z)=\frac{\operatorname{Im}h}{m_{\infty,\frac{1}{2}}(z)+\operatorname{Re}h}=-\frac{1}{i\sqrt{2z}+1}. \tag{11.9}$$

Direct substitution confirms that $V_{\Theta_{\infty,h_0}}(i)=i$. Clearly, $V_{\Theta_{\infty,h_0}}(z)\in\mathfrak{M}$ and our L-system $\Theta_{\infty,0}$ has infinite c-entropy according to Remark 23.

Similarly we can consider L-systems for which $h\neq -m_{\infty,\frac{1}{2}}(i)$ and hence $\kappa\neq 0$. Then according to (4.14) we have

$$\begin{aligned}\kappa&=\left|\frac{m_{\infty,\frac{1}{2}}(i)+h}{m_{\infty,\frac{1}{2}}(i)+\bar{h}}\right|=\left|\frac{\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i+h}{\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i+\bar{h}}\right|=\left|\frac{1-i+\sqrt{2}\operatorname{Re}h+\sqrt{2}\operatorname{Im}h\,i}{1-i+\sqrt{2}\operatorname{Re}h-\sqrt{2}\operatorname{Im}h\,i}\right|\\&=\sqrt{\frac{(1+\sqrt{2}\operatorname{Re}h)^2+(\sqrt{2}\operatorname{Im}h-1)^2}{(1+\sqrt{2}\operatorname{Re}h)^2+(\sqrt{2}\operatorname{Im}h+1)^2}}.\end{aligned}$$

If we want our operator T_h to be extremal, we (according to Theorem 11) set $\operatorname{Re}h=-m_{\infty,\frac{1}{2}}(-0)=0$. Then by the above calculations

$$\kappa_{ext}=\sqrt{\frac{1+(\sqrt{2}\operatorname{Im}h-1)^2}{1+(\sqrt{2}\operatorname{Im}h+1)^2}}=\sqrt{\frac{1+(\operatorname{Im}h)^2-\sqrt{2}\operatorname{Im}h}{1+(\operatorname{Im}h)^2+\sqrt{2}\operatorname{Im}h}},$$

where κ_{ext} is the von Neumann parameter of the operator T_h. As we have established in Sect. 5, if the main operator T_h is extremal, then κ_{ext} satisfies $\kappa_0\leq\kappa_{ext}<1$, where κ_0 is given by (5.9). In our case $m_{\infty,\frac{1}{2}}(-0)=0$ and $|m_{\infty,\frac{1}{2}}(i)|=1$, and hence

$$\kappa_0=\sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}}=\sqrt{2}-1. \tag{11.10}$$

Let us describe L-systems having the main extremal operator with von Neumann's parameter κ_0 as in (11.10). The corresponding to κ_0 value of h is (see (9.8))

$$\begin{aligned}h&=-m_{\infty,\frac{1}{2}}(-0)+i\sqrt{(\operatorname{Re}m_{\infty}(i)-m_{\infty,\frac{1}{2}}(-0))^2+(\operatorname{Im}m_{\infty,\frac{1}{2}}(i))^2}\\&=i|m_{\infty}(i)|=i.\end{aligned}$$

We have

$$\begin{cases} T_i\, y = -y'' \\ y'(1) = iy(1). \end{cases} \tag{11.11}$$

The quasi-kernel of the real part of the state-space operator of our family of L-systems is determined by (4.6) as follows

$$\begin{cases} \hat{A}_\mu\, y = -y'' \\ y'(1) = -\frac{1}{\mu} y(1). \end{cases} \tag{11.12}$$

Now we construct two L-systems having T_i as their main operator and attaining maximal c-entropy. We note that in the case in question the coefficient of dissipation $\mathcal{D} = 1$ and $\operatorname{Im} m_{\infty,\frac{1}{2}}(i) = -1/\sqrt{2}$. First we use (9.22) to obtain $\mu_1 = -1$. We get

$$\begin{cases} \hat{A}_{-1}\, y = -y'' \\ y'(1) = y(1). \end{cases}$$

The state-space operator of the L-system $\Theta_{-1,i}$ with κ_0 from (11.10) and $\mu = -1$ is (see (4.3))

$$\begin{aligned} \mathbb{A}_{-1,i}\, y &= -y'' - \frac{1}{1+i}\,[y'(1) - iy(1)]\,[\delta(x-1) - \delta'(x-1)], \\ \mathbb{A}^*_{-1,i}\, y &= -y'' - \frac{1}{1-i}\,[y'(1) + iy(1)]\,[\delta(x-1) - \delta'(x-1)]. \end{aligned} \tag{11.13}$$

Also the channel operator is given by $K_{-1,i}\, c = cg_{-1,i}$, $(c \in \mathbb{C})$, where (see (4.4))

$$g_{-1,i} = \sqrt{2}[\delta'(x-1) - \delta(x-1)].$$

Then $\Theta_{-1,i}$ has the form

$$\Theta_{-1,i} = \begin{pmatrix} \mathbb{A}_{-1,i} & K_{-1,i} & 1 \\ \mathcal{H}_+ \subset L_2[1,+\infty) \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix}, \tag{11.14}$$

where all the entries are described above. Using formulas (4.9) and (4.10) we obtain

$$W_{\Theta_{-1,i}}(z) = \frac{-1-i}{-1+i} \cdot \frac{m_{\infty,\frac{1}{2}}(z) + \overline{h}}{m_{\infty,\frac{1}{2}}(z) + h} = i\,\frac{\sqrt{z}+1}{\sqrt{z}-1} \tag{11.15}$$

and

$$V_{\Theta_{-1,i}}(z) = \frac{-i\sqrt{z}-1}{i\sqrt{z}-1} = -\frac{\sqrt{z}-i}{\sqrt{z}+i}. \tag{11.16}$$

Direct substitution yields

$$V_{\Theta_{-1,i}}(i) = -\frac{\sqrt{i}-i}{\sqrt{i}+i} = -\frac{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i-i}{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i+i} = (\sqrt{2}-1)i.$$

Clearly, $V_{\Theta_{-1,i}}(z) \in \mathfrak{M}_{\kappa_0}$, where κ_0 is given by (11.10). The L-system $\Theta_{-1,i}$ in (11.14) exemplifies the first L-system described in Remark 30.

In order to construct the second L-system described in Remark 30 we apply (9.23) and obtain $\mu_2 = 1$ first. Then we get

$$\begin{cases} \hat{A}_1\, y = -y'' \\ y'(1) = -y(1). \end{cases}$$

The state-space operator of the L-system $\Theta_{1,i}$ with κ_0 from (11.10) and $\mu = 1$ is (see (4.3))

$$\begin{aligned} \mathbb{A}_{1,i}\, y &= -y'' - \frac{1}{1-i}\,[y'(1) - iy(1)]\,[\delta(x-1)+\delta'(x-1)], \\ \mathbb{A}^*_{1,i}\, y &= -y'' - \frac{1}{1+i}\,[y'(1) + iy(1)]\,[\delta(x-1)+\delta'(x-1)]. \end{aligned} \tag{11.17}$$

Also the channel operator is given by $K_{1,i}\, c = cg_{1,i}$, $(c \in \mathbb{C})$, where (see (4.4))

$$g_{1,i} = \sqrt{2}[\delta(x-1)+\delta'(x-1)].$$

Then $\Theta_{1,i}$ has the form

$$\Theta_{1,i} = \begin{pmatrix} \mathbb{A}_{1,i} & K_{1,i} & 1 \\ \mathcal{H}_+ \subset L_2[1,+\infty) \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix}, \tag{11.18}$$

where all the components are described above. Using formulas (4.9) and (4.10) we obtain

$$W_{\Theta_{1,i}}(z) = \frac{1-i}{1+i}\cdot\frac{m_{\infty,\frac{1}{2}}(z)+\bar{h}}{m_{\infty,\frac{1}{2}}(z)+h} = -i\,\frac{\sqrt{z}+1}{\sqrt{z}-1} \tag{11.19}$$

and

$$V_{\Theta_{1,i}}(z) = \frac{\sqrt{z}+i}{\sqrt{z}-i}. \tag{11.20}$$

Hence

$$V_{\Theta_{1,i}}(i) = \frac{\sqrt{i}+i}{\sqrt{i}-i} = \frac{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i+i}{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i-i} = (\sqrt{2}+1)i.$$

Clearly, $V_{\Theta_{1,i}}(z) \in \mathfrak{M}_{\kappa_0}^{-1}$, where κ_0 is given by (11.10). The L-system $\Theta_{1,i}$ in (11.18) exemplifies the second L-system described in Remark 30.

Example 2

Let $\nu = 3/2$. It is known [2] that in this case

$$m_{\infty,\frac{3}{2}}(z) = -\frac{iz - \frac{3}{2}\sqrt{z} - \frac{3}{2}i}{\sqrt{z}+i} - \frac{1}{2} = \frac{\sqrt{z} - iz + i}{\sqrt{z}+i} = 1 - \frac{iz}{\sqrt{z}+i}$$

and

$$m_{\infty,\frac{3}{2}}(-0) = 1, \qquad m_{\infty,\frac{3}{2}}(i) = 1 + \frac{1}{\sqrt{2}} - \frac{i}{2} = \frac{2+\sqrt{2}-i}{2}.$$

The minimal symmetric operator then becomes

$$\begin{cases} \dot{A}\, y = -y'' + \frac{2}{x^2} y \\ y(1) = y'(1) = 0. \end{cases}$$

Note that in this case

$$\operatorname{Re} m_{\infty,\frac{1}{2}}(i) = 1 + \frac{1}{\sqrt{2}} > m_{\infty,\frac{1}{2}}(0) = 1,$$

which taking into account that the symmetric operator $\dot{A}$ is nonnegative, illustrates the result of Lemma 17.

We are going to construct an L-system with an extremal main operator and maximum c-entropy. In order to do that we take h defined by (9.18). Then

$$h = -1 + i\sqrt{\left(1 + \frac{1}{\sqrt{2}} - 1\right)^2 + \frac{1}{4}} = -1 + \frac{\sqrt{3}}{2} i. \tag{11.21}$$

Then the corresponding main operator T_h is

$$\begin{cases} T_h\, y = -y'' + \frac{2}{x^2} y \\ y'(1) = \left(-1 + \frac{\sqrt{3}}{2} i\right) y(1). \end{cases} \tag{11.22}$$

This value of h in (11.21) corresponds to the value of κ_0 given by (5.9) that is in this case

$$\kappa_0 = \frac{\sqrt{2}}{\sqrt{3}+1}. \tag{11.23}$$

Having in mind the result of Theorem 33, we would like to have an L-system with maximum c-entropy and accretive extremal state-space operator. To do so we take $\mu = \infty$ and construct the corresponding L-system $\Theta_{h,\infty}$. According to (4.6) the quasi-kernel in this case is

$$\begin{cases} \hat{A}_\infty\, y = -y'' + \frac{2}{x^2} y \\ y'(1) = -y(1), \end{cases}$$

which is the Krein-von Neumann extension of $\dot{A}$. The state-space operator of this L-system $\Theta_{h,\infty}$ with $\kappa = \kappa_0$ and $\mu = \infty$ is (see (4.3))

$$\begin{aligned} \mathbb{A}_{\infty,h}\, y &= -y'' + \frac{2}{x^2} y - \left[y'(1) + \left(1 - \frac{\sqrt{3}}{2} i\right) y(1) \right] \delta(x-1), \\ \mathbb{A}^*_{\infty,h}\, y &= -y'' + \frac{2}{x^2} y - \left[y'(1) + \left(1 + \frac{\sqrt{3}}{2} i\right) y(1) \right] \delta(x-1). \end{aligned}$$

Its impedance function $V_{\Theta_{\infty,h}}(z)$ is such that

$$V_{\Theta_{\infty,h}}(i) = \frac{\operatorname{Im} h}{m_{\infty,\frac{3}{2}}(i) + \operatorname{Re} h} = \frac{\frac{\sqrt{3}}{2}}{\frac{2+\sqrt{2}-i}{2} - 1} = \frac{\sqrt{3}}{\sqrt{2} - i} = \sqrt{\frac{2}{3}} + \frac{1}{\sqrt{3}} i.$$

Now we are going to construct two L-systems having T_h from (11.22) as their main operator and such that their corresponding impedance functions are from the classes $\mathfrak{M}_{\kappa_0}$ and $\mathfrak{M}^{-1}_{\kappa_0}$ (as discussed in Remark 30), where κ_0 is given by (11.23). First find the normalizing parameter a that corresponds to κ_0,

$$a = \frac{1-\kappa_0}{1+\kappa_0} = \frac{1 - \frac{\sqrt{2}}{\sqrt{3}+1}}{1 + \frac{\sqrt{2}}{\sqrt{3}+1}} = \frac{\sqrt{3} - \sqrt{2} + 1}{\sqrt{3} + \sqrt{2} + 1} = \sqrt{3} - \sqrt{2}.$$

Then we use $V_{\Theta_{\mu_1,h}}(i) = ai$ and (4.9) for above value of a to obtain

$$V_{\Theta_{\mu_1,h}}(i) = \frac{\left(\frac{2+\sqrt{2}-i}{2} + \mu_1\right) \frac{\sqrt{3}}{2}}{(\mu_1 + 1)\left(\frac{2+\sqrt{2}-i}{2}\right) - \mu_1 - \frac{7}{4}} = (\sqrt{3} - \sqrt{2}) i.$$

Solving the above equation for μ_1 we get $\mu_1 = -\frac{2+\sqrt{3}}{2}$. This yields the quasi-kernel (see (4.6))

$$\begin{cases} \hat{A}_{\mu_1}\, y = -y'' + \frac{2}{x^2} y \\ y'(1) = (\frac{\sqrt{3}}{2} - 1) y(1). \end{cases}$$

The state-space operator of the L-system $\Theta_{\mu_1,h}$ with κ_0 from (11.10) and $\mu_1 = -\frac{2+\sqrt{3}}{2}$ is (see (4.3))

$$\begin{aligned}\mathbb{A}_{\mu_1,h}\, y &= -y'' + \frac{2y}{x^2} - \frac{2y'(1) + (1-\sqrt{3}i)y(1)}{2\sqrt{3}(1+i)}\\ &\quad\times\Big[(2+\sqrt{3})\delta(x-1) - 2\delta'(x-1)\Big],\\ \mathbb{A}^*_{\mu_1,h}\, y &= -y'' + \frac{2y}{x^2} - \frac{2y'(1) + (1+\sqrt{3}i)y(1)}{2\sqrt{3}(1-i)}\\ &\quad\times\Big[(2+\sqrt{3})\delta(x-1) - 2\delta'(x-1)\Big].\end{aligned}$$

Also the channel operator is given by $K_{\mu_1,h}\, c = cg_{\mu_1,h}$, $(c \in \mathbb{C})$, where (see (4.4))

$$g_{\mu_1,h} = \frac{1}{2\cdot 3^{1/4}}\Big[2\delta'(x-1) - (2+\sqrt{3})\delta(x-1)\Big].$$

Then $\Theta_{\mu_1,h}$ has the form

$$\Theta_{\mu_1,h} = \begin{pmatrix} \mathbb{A}_{\mu_1,h} & K_{\mu_1,h} & 1\\ \mathcal{H}_+ \subset L_2[1,+\infty) \subset \mathcal{H}_- & & \mathbb{C}\end{pmatrix}, \tag{11.24}$$

where all the components are described above. Using formulas (4.9) and (4.10) we obtain

$$W_{\Theta_{\mu_1,h}}(z) = \frac{-1-i}{-1+i}\cdot\frac{m_{\infty,\frac{3}{2}}(z)+\bar{h}}{m_{\infty,\frac{3}{2}}(z)+h} = (-i)\frac{2z+\sqrt{3z}+\sqrt{3}i}{2z-\sqrt{3z}-\sqrt{3}i}$$

and

$$V_{\Theta_{\mu_1,h}}(z) = \frac{\Big(m_{\infty,\frac{3}{2}}(z)+\mu_1\Big)\operatorname{Im} h}{(\mu_1 - \operatorname{Re} h)\, m_{\infty,\frac{3}{2}}(z) + \mu_1 \operatorname{Re} h - |h|^2} = \frac{2iz - \sqrt{3z} - i\sqrt{3}}{2iz + \sqrt{3z} + i\sqrt{3}}. \tag{11.25}$$

Direct substitution confirms that

$$V_{\Theta_{\mu_1,h}}(i) = \frac{2i\cdot i - \sqrt{3i} - i\sqrt{3}}{2i\cdot i + \sqrt{3i} + i\sqrt{3}} = (\sqrt{3}-\sqrt{2})i.$$

Clearly,

$$V_{\Theta_{\mu_1,h}}(z) \in \mathfrak{M}_{\kappa_0},$$

where κ_0 is given by (11.23).

Now we apply (9.23) and obtain

$$\mu_2 = \frac{\mu_1 \text{Re } h - |h|^2}{\mu_1 - \text{Re } h} = \frac{\frac{2+\sqrt{3}}{2} - \frac{7}{4}}{1 - \frac{2+\sqrt{3}}{2}} = \frac{\sqrt{3}-2}{2}.$$

This yields

$$\begin{cases} \hat{A}_{\mu_2}\, y = -y'' + \frac{2}{x^2} y \\ y'(1) = (\frac{\sqrt{3}}{2} + 1) y(1). \end{cases}$$

The state-space operator of the L-system $\Theta_{\mu_2,h}$ with κ_0 from (11.10) and $\mu_2 = \frac{\sqrt{3}-2}{2}$ is (see (4.3))

$$\begin{aligned} \mathbb{A}_{\mu_2,h}\, y &= -y'' - \frac{2y}{x^2} + \frac{2y'(1) + (1-\sqrt{3}i)y(1)}{2\sqrt{3}(1-i)} \\ &\quad \times \left[(\sqrt{3}-2)\delta(x-1) - 2\delta'(x-1)\right], \\ \mathbb{A}^*_{\mu_2,h}\, y &= -y'' - \frac{2}{x^2} y + \frac{2y'(1) + (1+\sqrt{3}i)y(1)}{2\sqrt{3}(1+i)} \\ &\quad \times \left[(\sqrt{3}-2)\delta(x-1) - 2\delta'(x-1)\right]. \end{aligned}$$

Also the channel operator is given by $K_{\mu_2,h}\, c = c g_{\mu_2,h}$, $(c \in \mathbb{C})$, where (see (4.4))

$$g_{\mu_2,h} = \frac{1}{2 \cdot 3^{1/4}} \left[2\delta'(x-1) + (\sqrt{3}-2)\delta(x-1)\right].$$

Then the L-system $\Theta_{\mu_2,h}$ has the form

$$\Theta_{\mu_2,h} = \begin{pmatrix} \mathbb{A}_{\mu_2,h} & K_{\mu_2,h} & 1 \\ \mathcal{H}_+ \subset L_2[1,+\infty) \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix}. \tag{11.26}$$

Using formulas (4.9) we obtain

$$W_{\Theta_{\mu_2,h}}(z) = \frac{1-i}{1+i} \cdot \frac{m_{\infty,\frac{3}{2}}(z) + \bar{h}}{m_{\infty,\frac{3}{2}}(z) + h} = (i) \frac{2z + \sqrt{3z} + \sqrt{3}i}{2z - \sqrt{3z} - \sqrt{3}i} \tag{11.27}$$

and

$$V_{\Theta_{\mu_2,h}}(z) = -\frac{1}{V_{\Theta_{\mu_1,h}}(z)} = -\frac{2iz + \sqrt{3z} + i\sqrt{3}}{2iz - \sqrt{3z} - i\sqrt{3}}. \tag{11.28}$$

So that

$$V_{\Theta_{\mu_2,h}}(i) = \frac{1}{\sqrt{3}-\sqrt{2}} i = (\sqrt{3}+\sqrt{2})i,$$

and therefore

$$V_{\Theta_{\mu_2,h}}(z) \in \mathfrak{M}_{\kappa_0}^{-1},$$

where κ_0 is given by (11.23).

Acknowledgements The second author was partially supported by the Simons collaboration grant 00061759 while preparing this paper.

Data availability Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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