Realization of inverse Stieltjes functions $(-m_\alpha(z))$ by Schrödinger L-systems

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Abstract. We study L-system realizations of the original Weyl-Titchmarsh functions $(-m_\alpha(z))$. In the case when the minimal symmetric Schrödinger operator is non-negative, we describe the Schrödinger L-systems that realize inverse Stieltjes functions $(-m_\alpha(z))$. This approach allows to derive a necessary and sufficient conditions for the functions $(-m_\alpha(z))$ to be inverse Stieltjes. In particular, the criteria when $(-m_\infty(z))$ is an inverse Stieltjes function is provided. Moreover, the value $m_\infty(-0)$ and parameter α allow us to describe the geometric structure of the realizing $(-m_\alpha(z))$ L-system. Additionally, we present the conditions in terms of the parameter α when the main and associated operators of a realizing $(-m_\alpha(z))$ L-system have the same or different angle of sectoriality which sets connections with the Kato problem on sectorial extensions of sectorial forms.

Contents

1. Introduction

The current paper is the third part of the project (started in [7] and continued in [6]) that studies the realizations of the original Weyl-Titchmarsh function $m_\infty(z)$ and its linear-fractional transformation $m_\alpha(z)$ associated with a Schrödinger operator. We investigate the Herglotz-Nevanlinna functions $-m_\infty(z)$ and $1/m_\infty(z)$ as well as $-m_\alpha(z)$ and $1/m_\alpha(z)$ that are realized as impedance functions of L-systems containing a dissipative Schrödinger main operator T_h, $(\operatorname{Im} h > 0)$. These L-systems

2020 *Mathematics Subject Classification.* Primary 47A10; Secondary 47N50, 81Q10.

Key words and phrases. L-system, Schrödinger operator, transfer function, impedance function, Herglotz-Nevanlinna function, inverse Stieltjes function, Weyl-Titchmarsh function.

will be refer to as *Schrödinger L-systems* for the rest of the paper. All formal definitions and expositions of general and Schrödinger L-systems are given in Sections 2 and 4. Note that all Schrödinger L-systems $\Theta_{\mu,h}$ form a two-parametric family whose members are uniquely defined by a real-valued parameter μ and a complex boundary value h ($\operatorname{Im} h > 0$) of the main dissipative operator.

In this paper we concentrate on the case when the realizing Schrödinger L-systems are based on non-negative symmetric Schrödinger operator and have accretive main and *accumulative* state-space operator.[1] It was shown in [**1**] (see also [**8**]) that the impedance functions of L-systems with accumulative state-space operators are *inverse Stieltjes* functions. Following our approach from [**6**] here we also set focus on the situation when the realizing accumulative Schrödinger L-systems are sectorial (see Section 2 for the definition) and the functions $(-m_\alpha(z))$ are the members of *sectorial classes* $S^{-1,\beta}$ and S^{-1,β_1,β_2} of inverse Stieltjes functions that are described in Section 3. Section 5 is dedicated to the general realization results from [**7**] for the functions $(-m_\infty(z))$, $1/m_\infty(z)$, and $(-m_\alpha(z))$. In particular, we recall there that $(-m_\infty(z))$, $1/m_\infty(z)$, and $(-m_\alpha(z))$ can be realized as the impedance function of Schrödinger L-systems $\Theta_{0,i}$, $\Theta_{\infty,i}$, and $\Theta_{\tan\alpha,i}$, respectively.

Section 6 contains the main results of the paper when the realization results from Section 5 are applied to Schrödinger L-systems with non-negative symmetric Schrödinger operator to obtain important additional properties. Remark 7 of Section 6 provides us with the set of criteria for the functions $(-m_\alpha(z))$ to be Stiejtjes or inverse Stijeltjes. In particular, the Theorem 6 and Remark 7 give the necessary and sufficient conditions for $(-m_\infty(z))$ to be an inverse Stieltjes function. Using the results provided in Section 4, we obtain new properties of L-systems $\Theta_{\tan\alpha,i}$ whose impedance function belong to certain sectorial classes of inverse Stieltjes functions. We emphasize that these results are formulated in terms of the parameter α defining the function $m_\alpha(z)$. Also, the knowledge of the limit value $m_\infty(-0)$ and the value of parameter α lets us find the exact angles of sectoriality of the main T_i and associate $\tilde{\mathbb{A}}$ operators of a realizing L-system that establishes the connection to Kato's problem about sectorial extension of sectorial forms.

We conclude the paper with providing an example that illustrates the main concepts. All the results obtained in this article contribute to a further development of the theory of open physical systems conceived by M. Livšic in [**21**].

2. Preliminaries

For a pair of Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ we denote by $[\mathcal{H}_1,\mathcal{H}_2]$ the set of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. Let $\dot A$ be a closed, densely defined, symmetric operator in a Hilbert space $\mathcal{H}$ with inner product (f,g), $f,g\in\mathcal{H}$. Any non-symmetric operator T in $\mathcal{H}$ such that

$$\dot A\subset T\subset \dot A^*$$

is called a *quasi-self-adjoint extension* of $\dot A$.

Consider the rigged Hilbert space (see [**13**], [**1**]) $\mathcal{H}_+\subset\mathcal{H}\subset\mathcal{H}_-$, where $\mathcal{H}_+=\operatorname{Dom}(\dot A^*)$ and

$$(f,g)_+=(f,g)+(\dot A^* f,\dot A^* g),\quad f,g\in\operatorname{Dom}(A^*). \tag{1}$$

[1]The situation when the state-space operator of the realizing Schrödinger L-system was accretive was thoroughly considered in [**6**].

Let $\mathcal{R}$ be the *Riesz-Berezansky operator* $\mathcal{R}$ (see [13], [1]) which maps $\mathcal{H}_-$ onto $\mathcal{H}_+$ such that $(f,g)=(f,\mathcal{R}g)_+$ ($\forall f\in\mathcal{H}_+,\ g\in\mathcal{H}_-$) and $\|\mathcal{R}g\|_+=\|g\|_-$. Note that identifying the space conjugate to $\mathcal{H}_\pm$ with $\mathcal{H}_\mp$, we get that if $\mathbb{A}\in[\mathcal{H}_+,\mathcal{H}_-]$, then $\mathbb{A}^*\in[\mathcal{H}_+,\mathcal{H}_-]$. An operator $\mathbb{A}\in[\mathcal{H}_+,\mathcal{H}_-]$ is called a *self-adjoint bi-extension* of a symmetric operator $\dot{A}$ if $\mathbb{A}=\mathbb{A}^*$ and $\mathbb{A}\supset\dot{A}$. Let $\mathbb{A}$ be a self-adjoint bi-extension of $\dot{A}$ and let the operator $\hat{A}$ in $\mathcal{H}$ be defined as follows:

$$\mathrm{Dom}(\hat{A})=\{f\in\mathcal{H}_+:\mathbb{A}f\in\mathcal{H}\},\quad \hat{A}=\mathbb{A}\upharpoonright\mathrm{Dom}(\hat{A}).$$

The operator $\hat{A}$ is called a *quasi-kernel* of a self-adjoint bi-extension $\mathbb{A}$ (see [28], [1, Section 2.1]). According to the von Neumann Theorem (see [1, Theorem 1.3.1]) the domain of $\hat{A}$, a self-adjoint extension of $\dot{A}$, can be expressed as

$$\mathrm{Dom}(\hat{A})=\mathrm{Dom}(\dot{A})\oplus(I+U)\mathfrak{N}_i,$$

where von Neumann's parameter U is a $(\cdot)$ (and $(+)$)-isometric operator from $\mathfrak{N}_i$ into $\mathfrak{N}_{-i}$ and

$$\mathfrak{N}_{\pm i}=\mathrm{Ker}\,(\dot{A}^*\mp iI)$$

are the deficiency subspaces of $\dot{A}$.

A self-adjoint bi-extension $\mathbb{A}$ of a symmetric operator $\dot{A}$ is called *t-self-adjoint* (see [1, Definition 4.3.1]) if its quasi-kernel $\hat{A}$ is self-adjoint operator in $\mathcal{H}$. An operator $\mathbb{A}\in[\mathcal{H}_+,\mathcal{H}_-]$ is called a *quasi-self-adjoint bi-extension* of an operator T if $\mathbb{A}\supset T\supset\dot{A}$ and $\mathbb{A}^*\supset T^*\supset\dot{A}$.

We are mostly interested in the following type of quasi-self-adjoint bi-extensions. Let T be a quasi-self-adjoint extension of $\dot{A}$ with nonempty resolvent set $\rho(T)$. A quasi-self-adjoint bi-extension $\mathbb{A}$ of an operator T is called (see [1, Definition 3.3.5]) a *$(*)$-extension* of T if $\mathrm{Re}\,\mathbb{A}$ is a t-self-adjoint bi-extension of $\dot{A}$. In what follows we assume that $\dot{A}$ has deficiency indices $(1,1)$. In this case it is known [1] that every quasi-self-adjoint extension T of $\dot{A}$ admits $(*)$-extensions. The description of all $(*)$-extensions via Riesz-Berezansky operator $\mathcal{R}$ can be found in [1, Section 4.3].

Recall that a linear operator T in a Hilbert space $\mathcal{H}$ is called **accretive** [19] if $\mathrm{Re}\,(Tf,f)\ge 0$ for all $f\in\mathrm{Dom}(T)$. We call an accretive operator T **β-sectorial** [19] if there exists a value of $\beta\in(0,\pi/2)$ such that

$$(\cot\beta)|\,\mathrm{Im}(Tf,f)|\le\ \mathrm{Re}\,(Tf,f),\qquad f\in\mathrm{Dom}(T). \tag{2}$$

We say that the angle of sectoriality β is **exact** for a β-sectorial operator T if

$$\tan\beta=\sup_{f\in\mathrm{Dom}(T)}\frac{|\,\mathrm{Im}(Tf,f)|}{\mathrm{Re}\,(Tf,f)}.$$

An accretive operator is called **extremal accretive** if it is not β-sectorial for any $\beta\in(0,\pi/2)$. A $(*)$-extension $\mathbb{A}$ of T is called **accretive** if $\mathrm{Re}\,(\mathbb{A}f,f)\ge 0$ for all $f\in\mathcal{H}_+$. This is equivalent to that the real part $\mathrm{Re}\,\mathbb{A}=(\mathbb{A}+\mathbb{A}^*)/2$ is a nonnegative t-self-adjoint bi-extension of $\dot{A}$.

A $(*)$-extensions $\mathbb{A}$ of an operator T is called **accumulative** (see [1]) if

$$(\mathrm{Re}\,\mathbb{A}f,f)\le(\dot{A}^*f,f)+(f,\dot{A}^*f),\quad f\in\mathcal{H}_+. \tag{3}$$

The definition below is a "lite" version of the definition of L-system given for a scattering L-system with one-dimensional input-output space. It is tailored for the case when the symmetric operator of an L-system has deficiency indices $(1,1)$. The general definition of an L-system can be found in [1, Definition 6.3.4] (see also [11] for a non-canonical version).

Definition 1. *An array*

$$\Theta = \begin{pmatrix} \mathbb{A} & K & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix} \tag{4}$$

*is called an **L-system** if:*

1. *T is a dissipative ($\operatorname{Im}(Tf,f) \geq 0$, $f \in \operatorname{Dom}(T)$) quasi-self-adjoint extension of a symmetric operator $\dot{A}$ with deficiency indices $(1,1)$;*
2. *$\mathbb{A}$ is a $(*)$-extension of T;*
3. *$\operatorname{Im}\mathbb{A} = KK^*$, where $K \in [\mathbb{C}, \mathcal{H}_-]$ and $K^* \in [\mathcal{H}_+, \mathbb{C}]$.*

Operators T and $\mathbb{A}$ are called a *main and state-space operators respectively* of the system Θ, and K is a *channel operator*. It is easy to see that the operator $\mathbb{A}$ of the system (4) is such that $\operatorname{Im}\mathbb{A} = (\cdot,\chi)\chi$, $\chi \in \mathcal{H}_-$ and pick $Kc = c\cdot\chi$, $c \in \mathbb{C}$ (see [**1**]). A system Θ in (4) is called *minimal* if the operator $\dot{A}$ is a prime operator in $\mathcal{H}$, i.e., there exists no non-trivial reducing invariant subspace of $\mathcal{H}$ on which it induces a self-adjoint operator. Minimal L-systems of the form (4) with one-dimensional input-output space were also considered in [**5**].

We associate with an L-system Θ the function

$$W_\Theta(z) = I - 2iK^*(\mathbb{A} - zI)^{-1}K, \quad z \in \rho(T), \tag{5}$$

which is called the **transfer function** of the L-system Θ. We also consider the function

$$V_\Theta(z) = K^*(\operatorname{Re}\mathbb{A} - zI)^{-1}K, \tag{6}$$

that is called the **impedance function** of an L-system Θ of the form (4). The transfer function $W_\Theta(z)$ of the L-system Θ and function $V_\Theta(z)$ of the form (6) are connected by the following relations valid for $\operatorname{Im} z \neq 0$, $z \in \rho(T)$,

$$\begin{aligned} V_\Theta(z) &= i[W_\Theta(z) + I]^{-1}[W_\Theta(z) - I], \\ W_\Theta(z) &= (I + iV_\Theta(z))^{-1}(I - iV_\Theta(z)). \end{aligned}$$

We say that an L-system Θ of the form (4) is called an **accretive L-system** ([**10**], [**16**]) if its state-space operator operator $\mathbb{A}$ is accretive, that is $\operatorname{Re}(\mathbb{A}f,f) \geq 0$ for all $f \in \mathcal{H}_+$, and **accumulative** ([**9**]) if its state-space operator $\mathbb{A}$ is accumulative, i.e., satisfies (3). It is easy to see that if an L-system is accumulative, then (3) implies that the operator $\dot{A}$ of the system is non-negative and both operators T and T^* are accretive. We also associate another operator $\tilde{\mathbb{A}}$ to an accumulative L-system Θ. It is given by

$$\tilde{\mathbb{A}} = 2\operatorname{Re}\dot{A}^* - \mathbb{A}, \tag{7}$$

where $\dot{A}^*$ is in $[\mathcal{H}_+, \mathcal{H}_-]$. Obviously, $\operatorname{Re}\dot{A}^* \in [\mathcal{H}_+, \mathcal{H}_-]$ and $\tilde{\mathbb{A}} \in [\mathcal{H}_+, \mathcal{H}_-]$. Clearly, $\tilde{\mathbb{A}}$ is a bi-extension of $\dot{A}$ and is accretive if and only if $\mathbb{A}$ is accumulative. It is also not hard to see that even though $\tilde{\mathbb{A}}$ is not a $(*)$-extensions of the operator T but the form $(\tilde{\mathbb{A}}f,f)$, $f \in \mathcal{H}_+$ extends the form (f,Tf), $f \in \operatorname{Dom}(T)$. An accretive L-system is called **sectorial** if the operator $\mathbb{A}$ is sectorial, i.e., satisfies (2) for some $\beta \in (0,\pi/2)$ and all $f \in \mathcal{H}_+$. Similarly, an accumulative L-system is **sectorial** if its operator $\tilde{\mathbb{A}}$ of the form (7) is sectorial.

3. Sectorial classes of inverse Stieltjes functions

It is known that a scalar function $V(z)$ is called the Herglotz-Nevanlinna function if it is holomorphic on $\mathbb{C}\setminus\mathbb{R}$, symmetric with respect to the real axis, i.e., $V(z)^* = V(\bar z)$, $z\in\mathbb{C}\setminus\mathbb{R}$, and if it satisfies the positivity condition $\operatorname{Im} V(z)\ge 0$, $z\in\mathbb{C}_+$. A complete description of the class of all Herglotz-Nevanlinna functions, that can be realized as impedance functions of L-systems can be found in [**1**], [**5**], [**15**], [**17**]. A scalar Herglotz-Nevanlinna function $V(z)$ is a *Stieltjes function* (see [**18**]) if it is holomorphic in $\operatorname{Ext}[0,+\infty)$ and

$$\frac{\operatorname{Im}[zV(z)]}{\operatorname{Im} z}\ge 0. \tag{8}$$

Now we turn to the definition of inverse Stieltjes functions (see [**18**], [**1**]). A scalar Herglotz-Nevanlinna function $V(z)$ is called **inverse Stieltjes** if $V(z)$ it is holomorphic in $\operatorname{Ext}[0,+\infty)$ and

$$\frac{\operatorname{Im}[V(z)/z]}{\operatorname{Im}\, z}\ge 0. \tag{9}$$

We will consider the inverse Stieltjes function $V(z)$ that admit (see [**18**]) the following integral representation

$$V(z)=\gamma+\int_0^\infty\left(\frac{1}{t-z}-\frac{1}{t}\right)dG(t), \tag{10}$$

where $\gamma\le 0$ and $G(t)$ is a non-decreasing on $[0,+\infty)$ function such that $\int_0^\infty\frac{dG(t)}{t+t^2}<\infty$. The following definition provides the description of a realizable subclass of inverse Stieltjes functions. A scalar inverse Stieltjes function $V(z)$ is a member of the **class** $S_0^{-1}(R)$ if the measure $G(t)$ in representation (10) is unbounded.It was shown in [**1**, Section 9.9] that a function $V(z)$ belongs to the class $S_0^{-1}(R)$ if and only if it can be realized as impedance function of an accumulative L-system Θ of the form (4) with a non-negative densely defined symmetric operator $\dot A$.

The definition of **sectorial subclasses** $S^{-1,\beta}$ of scalar inverse Stieltjes functions is the following. An inverse Stieltjes function $V(z)$ belongs to $S^{-1,\beta}$ if

$$K_\beta=\sum_{k,l=1}^n\left[\frac{V(z_k)/z_k-V(\bar z_l)/\bar z_l}{z_k-\bar z_l}-(\cot\beta)\,\frac{V(\bar z_l)}{\bar z_l}\frac{V(z_k)}{z_k}\right]h_k\bar h_l\ge 0, \tag{11}$$

for an arbitrary sequences of complex numbers $\{z_k\}$, $(\operatorname{Im} z_k>0)$ and $\{h_k\}$, $(k=1,\dots,n)$. For $0<\beta_1<\beta_2<\frac{\pi}{2}$, we have

$$S^{-1,\beta_1}\subset S^{-1,\beta_2}\subset S^{-1},$$

where S^{-1} denotes the class of all inverse Stieltjes functions (which corresponds to the case $\beta=\frac{\pi}{2}$).

Let Θ be an accumulative minimal L-system of the form (4). It was shown in [**12**] that the impedance function $V_\Theta(z)$ defined by (6) belongs to the class $S^{-1,\beta}$ if and only if the operator $\tilde{\mathbb{A}}$ of the form (7) associated to the L-system Θ is β-sectorial.

Let $0\le\beta_1<\frac{\pi}{2}$, $0<\beta_2\le\frac{\pi}{2}$, and $\beta_1\le\beta_2$. We say that a scalar inverse Stieltjes function $V(z)$ of the class $S_0^{-1}(R)$ belongs to the **class** S^{-1,β_1,β_2} if

$$\tan(\pi-\beta_1)=\lim_{x\to 0}V(x),\qquad \tan(\pi-\beta_2)=\lim_{x\to-\infty}V(x). \tag{12}$$

The following connection between the classes $S^{-1,\beta}$ and S^{-1,β_1,β_2} was established in [**12**]. Let Θ be an accumulative L-system of the form (4) with a densely defined non-negative symmetric operator $\dot{A}$. Let also $\tilde{\mathbb{A}}$ of the form (7) be β-sectorial. Then the impedance function $V_\Theta(z)$ defined by (6) belongs to the class S^{-1,β_1,β_2}. Moreover, the operator T of Θ is $(\beta_2-\beta_1)$-sectorial with the exact angle of sectoriality $(\beta_2-\beta_1)$, and $\tan\beta_2 \le \tan\beta$. Note, that this also remains valid for the case when the operator $\tilde{\mathbb{A}}$ is accretive but not β-sectorial for any $\beta \in (0,\pi/2)$. It also follows that under the same set of assumptions, if β is the exact angle of sectoriality of the operator T, then $V_\Theta(z) \in S^{-1,0,\beta}$ and is such that $\gamma = 0$ in (10).

Let Θ be a minimal accumulative L-system of the form (4) as above. Let also $\tilde{\mathbb{A}}$ be defined via (7). It was shown in [**12**] that if the impedance function $V_\Theta(z)$ belongs to the class S^{-1,β_1,β_2} and $\beta_2 \neq \pi/2$, then $\tilde{\mathbb{A}}$ is β-sectorial, where $\tan\beta$ is defined via

$$\tan\beta = \tan\beta_2 + 2\sqrt{\tan\beta_1(\tan\beta_2 - \tan\beta_1)}. \tag{13}$$

Moreover, both $\tilde{\mathbb{A}}$ and T are β-sectorial operators with the exact angle $\beta \in (0,\pi/2)$ if and only if $V_\Theta(z) \in S^{-1,0,\beta}$ and

$$\tan\beta = \int_0^\infty \frac{dG(t)}{t}, \tag{14}$$

where $G(t)$ is the measure from integral representation (10) of $V_\Theta(z)$ (see [**12**, Theorem 13]).

4. Construction of a Schrödinger L-system

Consider $\mathcal{H} = L_2[\ell,+\infty)$, $\ell \ge 0$, and $l(y) = -y'' + q(x)y$, where q is a real locally summable on $[\ell,+\infty)$ function. Suppose that the symmetric operator

$$\begin{cases} \dot{A}y = -y'' + q(x)y \\ y(\ell) = y'(\ell) = 0 \end{cases} \tag{15}$$

has deficiency indices (1,1). Let D^* be the set of functions locally absolutely continuous together with their first derivatives such that $l(y) \in L_2[\ell,+\infty)$. Consider $\mathcal{H}_+ = \mathrm{Dom}(\dot{A}^*) = D^*$ with the scalar product

$$(y,z)_+ = \int_\ell^\infty \left(y(x)\overline{z(x)} + l(y)\overline{l(z)}\right)dx, \quad y,\ z \in D^*.$$

Let $\mathcal{H}_+ \subset L_2[\ell,+\infty) \subset \mathcal{H}_-$ be the corresponding triplet of Hilbert spaces and the operators T_h and T_h^* are

$$\begin{cases} T_h y = l(y) = -y'' + q(x)y \\ hy(\ell) - y'(\ell) = 0 \end{cases}, \quad \begin{cases} T_h^* y = l(y) = -y'' + q(x)y \\ \overline{h}y(\ell) - y'(\ell) = 0 \end{cases}, \tag{16}$$

where $\mathrm{Im}\, h > 0$. Suppose $\dot{A}$ is a symmetric operator of the form (15) with deficiency indices (1,1), generated by the differential operation $l(y) = -y'' + q(x)y$. Let also $\varphi_k(x,\lambda)(k=1,2)$ be the solutions of the following Cauchy problems:

$$\begin{cases} l(\varphi_1) = \lambda\varphi_1 \\ \varphi_1(\ell,\lambda) = 0 \\ \varphi_1'(\ell,\lambda) = 1 \end{cases}, \qquad \begin{cases} l(\varphi_2) = \lambda\varphi_2 \\ \varphi_2(\ell,\lambda) = -1 \\ \varphi_2'(\ell,\lambda) = 0 \end{cases}.$$

It is well known [22], [20] that there exists a function $m_\infty(\lambda)$ introduced by H. Weyl [29] for which

$$\varphi(x,\lambda)=\varphi_2(x,\lambda)+m_\infty(\lambda)\varphi_1(x,\lambda)$$

belongs to $L_2[\ell,+\infty)$. It is important for our discussion that the function $m_\infty(\lambda)$ is not a Herglotz-Nevanlinna function but $(-m_\infty(\lambda))$ and $(1/m_\infty(\lambda))$ are (see [20], [22]).

A construction of an L-system associated with a non-self-adjoint Schrödinger operator T_h was thoroughly described in [1]. In particular, it was shown (see also [3]) that the set of all $(*)$-extensions of the non-self-adjoint Schrödinger operator T_h of the form (16) in $L_2(\ell,+\infty)$ is given by

$$\begin{aligned}\mathbb{A}_{\mu,h}\,y&=-y''+q(x)y-\frac{1}{\mu-h}\,[y'(\ell)-hy(\ell)]\,[\mu\delta(x-\ell)+\delta'(x-\ell)],\\ \mathbb{A}^*_{\mu,h}\,y&=-y''+q(x)y-\frac{1}{\mu-\overline{h}}\,[y'(\ell)-\overline{h}y(\ell)]\,[\mu\delta(x-\ell)+\delta'(x-\ell)].\end{aligned}\tag{17}$$

Note that the formulas (17) establish a one-to-one correspondence between the set of all $(*)$-extensions of a Schrödinger operator T_h of the form (16) and all real numbers $\mu\in[-\infty,+\infty]$. It is easy to check that the $(*)$-extension $\mathbb{A}$ in (17) satisfies the condition

$$\operatorname{Im}\mathbb{A}_{\mu,h}=\frac{\mathbb{A}_{\mu,h}-\mathbb{A}^*_{\mu,h}}{2i}=(.,g_{\mu,h})g_{\mu,h},$$

where

$$g_{\mu,h}=\frac{(\operatorname{Im}h)^{\frac12}}{|\mu-h|}\,[\mu\delta(x-\ell)+\delta'(x-\ell)]\tag{18}$$

and $\delta(x-\ell),\delta'(x-\ell)$ are the delta-function and its derivative at the point ℓ, respectively. Furthermore,

$$(y,g_{\mu,h})=\frac{(\operatorname{Im}h)^{\frac12}}{|\mu-h|}\;[\mu y(\ell)-y'(\ell)],$$

where $y\in\mathcal{H}_+$, $g_{\mu,h}\in\mathcal{H}_-$, and $\mathcal{H}_+\subset L_2[\ell,+\infty)\subset\mathcal{H}_-$ is the triplet of Hilbert spaces discussed above.

It was also shown in [1] that the quasi-kernel $\hat{A}_\xi$ of $\operatorname{Re}\mathbb{A}_{\mu,h}$ is given by

$$\left\{\begin{array}{l}\hat{A}_\xi y=-y''+q(x)y\\ y'(\ell)=\xi y(\ell)\end{array}\right.,\quad\text{where}\quad \xi=\frac{\mu\operatorname{Re}h-|h|^2}{\mu-\operatorname{Re}h}.\tag{19}$$

Take operator $K_{\mu,h}c=cg_{\mu,h}$, $(c\in\mathbb{C})$. Clearly,

$$K^*_{\mu,h}y=(y,g_{\mu,h}),\quad y\in\mathcal{H}_+,\tag{20}$$

and $\operatorname{Im}\mathbb{A}_{\mu,h}=K_{\mu,h}K^*_{\mu,h}$. Therefore,

$$\Theta_{\mu,h}=\begin{pmatrix}\mathbb{A}_{\mu,h}&K_{\mu,h}&1\\ \mathcal{H}_+\subset L_2[\ell,+\infty)\subset\mathcal{H}_-&&\mathbb{C}\end{pmatrix},\tag{21}$$

is an L-system with the main operator T_h, $(\operatorname{Im}h>0)$ of the form (16), the state-space operator $\mathbb{A}_{\mu,h}$ of the form (17), and with the channel operator $K_{\mu,h}$ of the form (20). In what follows we will refer to $\Theta_{\mu,h}$ as a *Schrödinger L-system.* It was established in [3], [1] that the transfer and impedance functions of $\Theta_{\mu,h}$ are

$$W_{\Theta_{\mu,h}}(z)=\frac{\mu-h}{\mu-\overline{h}}\;\frac{m_\infty(z)+\overline{h}}{m_\infty(z)+h},\tag{22}$$

and

$$V_{\Theta_{\mu,h}}(z)=\frac{(m_\infty(z)+\mu)\,\mathrm{Im}\,h}{(\mu-\mathrm{Re}\,h)\,m_\infty(z)+\mu\mathrm{Re}\,h-|h|^2}. \tag{23}$$

5. Schrödinger L-system realizations of $-m_\infty(z)$, $1/m_\infty(z)$ and $m_\alpha(z)$

As we have already mentioned in Section 4, the original Weyl-Titchmarsh function $m_\infty(z)$ has a property that $(-m_\infty(z))$ is a Herglotz-Nevanlinna function (see [20], [22]). A problem whether $(-m_\infty(z))$ can be realized as the impedance function of a Schrödinger L-system was solved in the following theorem proved in [7].

THEOREM 2 ([7]). *Let $\dot A$ be a symmetric Schrödinger operator of the form (15) with deficiency indices $(1,1)$ and locally summable potential in $\mathcal{H}=L^2[\ell,\infty)$. If $m_\infty(z)$ is the Weyl-Titchmarsh function of $\dot A$, then the Herglotz-Nevanlinna function $(-m_\infty(z))$ can be realized as the impedance function of a Schrödinger L-system $\Theta_{\mu,h}$ of the form (21) with $\mu=0$ and $h=i$.*

Conversely, let $\Theta_{\mu,h}$ be a Schrödinger L-system of the form (21) with the symmetric operator $\dot A$ such that $V_{\Theta_{\mu,h}}(z)=-m_\infty(z)$, for all $z\in\mathbb{C}_\pm$ and $\mu\in\mathbb{R}\cup\{\infty\}$. Then the parameters μ and h defining $\Theta_{\mu,h}$ are such that $\mu=0$ and $h=i$.

An analogues result for the function $1/m_\infty(z)$ also takes place (see [7]).

THEOREM 3 ([7]). *Let $\dot A$ be a symmetric Schrödinger operator of the form (15) with deficiency indices $(1,1)$ and locally summable potential in $\mathcal{H}=L^2[\ell,\infty)$. If $m_\infty(z)$ is the Weyl-Titchmarsh function of $\dot A$, then the Herglotz-Nevanlinna function $(1/m_\infty(z))$ can be realized as the impedance function of a Schrödinger L-system $\Theta_{\mu,h}$ of the form (21) with $\mu=\infty$ and $h=i$.*

Conversely, let $\Theta_{\mu,h}$ be a Schrödinger L-system of the form (21) with the symmetric operator $\dot A$ such that $V_{\Theta_{\mu,h}}(z)=\frac{1}{m_\infty(z)}$, for all $z\in\mathbb{C}_\pm$ and $\mu\in\mathbb{R}\cup\{\infty\}$. Then the parameters μ and h defining $\Theta_{\mu,h}$ are such that $\mu=\infty$ and $h=i$.

One can note that both L-systems $\Theta_{0,i}$ and $\Theta_{\infty,i}$ obtained in Theorems 2 and 3 share the same main operator

$$\begin{cases} T_i\, y=-y''+q(x)y\\ y'(\ell)=i\,y(\ell)\end{cases}. \tag{24}$$

The Weyl-Titchmarsh functions $m_\alpha(z)$ are defined as follows. Let $\dot A$ be a symmetric operator of the form (15) with deficiency indices (1,1), generated by the differential operation $l(y)=-y''+q(x)y$. Let also $\varphi_\alpha(x,z)$ and $\theta_\alpha(x,z)$ be the solutions of the following Cauchy problems:

$$\begin{cases} l(\varphi_\alpha)=z\varphi_\alpha\\ \varphi_\alpha(\ell,z)=\sin\alpha\\ \varphi'_\alpha(\ell,z)=-\cos\alpha\end{cases},\qquad \begin{cases} l(\theta_\alpha)=z\theta_\alpha\\ \theta_\alpha(\ell,z)=\cos\alpha\\ \theta'_\alpha(\ell,z)=\sin\alpha\end{cases}.$$

One can show [14], [22], [23] that there exists an analytic in $\mathbb{C}_\pm$ function $m_\alpha(z)$ for which

$$\psi(x,z)=\theta_\alpha(x,z)+m_\alpha(z)\varphi_\alpha(x,z) \tag{25}$$

belongs to $L_2[\ell,+\infty)$. It is easy to see that if $\alpha=\pi$, then $m_\pi(z)=m_\infty(z)$. The functions $m_\alpha(z)$ and $m_\infty(z)$ are connected (see [14], [23]) by

$$m_\alpha(z)=\frac{\sin\alpha+m_\infty(z)\cos\alpha}{\cos\alpha-m_\infty(z)\sin\alpha}. \tag{26}$$

It is known [22], [23] that for any real α the function $-m_\alpha(z)$ is a Herglotz-Nevanlinna function. Also, (26) yields

$$-m_\alpha(z) = \frac{\sin\alpha + m_\infty(z)\cos\alpha}{-\cos\alpha + m_\infty(z)\sin\alpha} = \frac{\cos\alpha + \frac{1}{m_\infty(z)}\sin\alpha}{\sin\alpha - \frac{1}{m_\infty(z)}\cos\alpha}. \tag{27}$$

The theorem below was proved in [7] for Herglotz-Nevanlinna functions $-m_\alpha(z)$ and is similar to Theorem 2.

Theorem 4 ([7]). *Let $\dot A$ be a symmetric Schrödinger operator of the form (15) with deficiency indices $(1,1)$ and locally summable potential in $\mathcal{H} = L^2[\ell,\infty)$. If $m_\alpha(z)$ is the function of $\dot A$ described in (25), then the Herglotz-Nevanlinna function $(-m_\alpha(z))$ can be realized as the impedance function of a Schrödinger L-system $\Theta_{\mu,h}$ of the form (21) with*

$$\mu = \tan\alpha \quad \textit{and} \quad h = i. \tag{28}$$

Conversely, let $\Theta_{\mu,h}$ be a Schrödinger L-system of the form (21) with the symmetric operator $\dot A$ such that

$$V_{\Theta_{\mu,h}}(z) = -m_\alpha(z),$$

for all $z \in \mathbb{C}_\pm$ and $\mu \in \mathbb{R}\cup\{\infty\}$. Then the parameters μ and h defining $\Theta_{\mu,h}$ are given by (28), i.e., $\mu = \tan\alpha$ and $h = i$.

Clearly, when $\alpha = \pi$ we obtain $\mu_\alpha = 0$, $m_\pi(z) = m_\infty(z)$, and the realizing L-system $\Theta_{0,i}$ is thoroughly described in [7, Section 5]. If $\alpha = \pi/2$, then we get $\mu_\alpha = \infty$, $-m_\alpha(z) = 1/m_\infty(z)$, and the realizing L-system is $\Theta_{\infty,i}$ (see [7, Section 5]). Excluding the cases when $\alpha = \pi$ or $\alpha = \pi/2$, we give the description of a Schrödinger L-system $\Theta_{\mu_\alpha,i}$ realizing $-m_\alpha(z)$ for $\alpha \in (0,\pi]$ as follows

$$\Theta_{\tan\alpha,i} = \begin{pmatrix} \mathbb{A}_{\tan\alpha,i} & K_{\tan\alpha,i} & 1\\ \mathcal{H}_+ \subset L_2[\ell,+\infty) \subset \mathcal{H}_- & & \mathbb{C}\end{pmatrix}, \tag{29}$$

where

$$\begin{aligned}\mathbb{A}_{\tan\alpha,i}\, y &= l(y) - \frac{1}{\tan\alpha - i}[y'(\ell) - iy(\ell)][(\tan\alpha)\delta(x-\ell) + \delta'(x-\ell)],\\ \mathbb{A}^*_{\tan\alpha,i}\, y &= l(y) - \frac{1}{\tan\alpha + i}\,[y'(\ell) + iy(\ell)][(\tan\alpha)\delta(x-\ell) + \delta'(x-\ell)],\end{aligned} \tag{30}$$

$K_{\tan\alpha,i}\, c = c\, g_{\tan\alpha,i}$, $(c \in \mathbb{C})$ and

$$g_{\tan\alpha,i} = (\tan\alpha)\delta(x-\ell) + \delta'(x-\ell). \tag{31}$$

It is also worth mentioning that

$$\begin{aligned}&V_{\Theta_{\tan\alpha,i}}(z) = -m_\alpha(z)\\ &W_{\Theta_{\tan\alpha,i}}(z) = \frac{\tan\alpha - i}{\tan\alpha + i}\cdot\frac{m_\infty(z) - i}{m_\infty(z) + i} = (-e^{2\alpha i})\,\frac{m_\infty(z) - i}{m_\infty(z) + i}.\end{aligned} \tag{32}$$

Similar to Theorem 3 results for the functions $1/m_\alpha(z)$ can be found in [7].

6. Accumulative Schrödinger L-systems

In this section we assume that $\dot{A}$ is a *non-negative* (i.e., $(\dot{A}f, f) \geq 0$ for all $f \in \mathrm{Dom}(\dot{A})$) symmetric operator of the form (15) with deficiency indices (1,1), generated by the differential operation $l(y) = -y'' + q(x)y$. The following theorem takes place.

Theorem 5 ([**25**], [**26**], [**27**]). *Let $\dot{A}$ be a nonnegative symmetric Schrödinger operator of the form (15) with deficiency indices $(1,1)$ and locally summable potential in $\mathcal{H} = L^2[\ell, \infty)$. Consider operator T_h of the form (16). Then*

(1) operator $\dot{A}$ has more than one non-negative self-adjoint extension, i.e., the Friedrichs extension A_F and the Kreĭn-von Neumann extension A_K do not coincide, if and only if $m_\infty(-0) < \infty$;
(2) operator T_h, $(h = \bar{h})$ coincides with the Kreĭn-von Neumann extension A_K if and only if $h = -m_\infty(-0)$;
(3) operator T_h is accretive if and only if

$$\mathrm{Re}\, h \geq -m_\infty(-0); \tag{33}$$

(4) operator T_h, $(h \neq \bar{h})$ is β-sectorial if and only if $\mathrm{Re}\, h > -m_\infty(-0)$ holds;
(5) operator T_h, $(h \neq \bar{h})$ is accretive but not β-sectorial for any $\beta \in (0, \frac{\pi}{2})$ if and only if $\mathrm{Re}\, h = -m_\infty(-0)$
(6) If $T_h, (\mathrm{Im}\, h > 0)$ is β-sectorial, then the exact angle β can be calculated via

$$\tan \beta = \frac{\mathrm{Im}\, h}{\mathrm{Re}\, h + m_\infty(-0)}. \tag{34}$$

In what follows, we assume that $m_\infty(-0) < \infty$. Then according to Theorem 5 (see also [**2**], [**24**], [**27**]) the operator T_h, $(\mathrm{Im}\, h > 0)$ of the form (16) is accretive and/or sectorial. If in this case T_h is accretive, then (see [**1**]) for all real μ satisfying the inequality

$$\mu \geq \frac{(\mathrm{Im}\, h)^2}{m_\infty(-0) + \mathrm{Re}\, h} + \mathrm{Re}\, h, \tag{35}$$

formulas (17) define the set of all accretive $(*)$-extensions $\mathbb{A}_{\mu,h}$ of T_h. Moreover, $\mathbb{A}_{\mu,h}$ is accretive but not β-sectorial for any $\beta \in (0, \pi/2)$ $(*)$-extension of T_h if and only if in (17)

$$\mu = \frac{(\mathrm{Im}\, h)^2}{m_\infty(-0) + \mathrm{Re}\, h} + \mathrm{Re}\, h, \tag{36}$$

(see [**8**, Theorem 4]). It is also shown in [**1**] that $(*)$-extensions $\mathbb{A}_{\mu,h}$ of the operator T_h are accumulative if and only if

$$-m_\infty(-0) \leq \mu \leq \mathrm{Re}\, h. \tag{37}$$

Using formulas (17) and direct calculations (see also [**8**]) one can obtain the formula for operator $\tilde{\mathbb{A}}_{\mu,h}$ of the form (7) as follows

$$\begin{aligned} \tilde{\mathbb{A}}_{\mu,h} y &= -y'' + q(x)y - y'(a)\delta(x-a) - y(a)\delta'(x-a) \\ &\quad + \frac{1}{\mu - h}\,[y'(a) - hy(a)]\,[\mu\delta(x-a) + \delta'(x-a)]. \end{aligned} \tag{38}$$

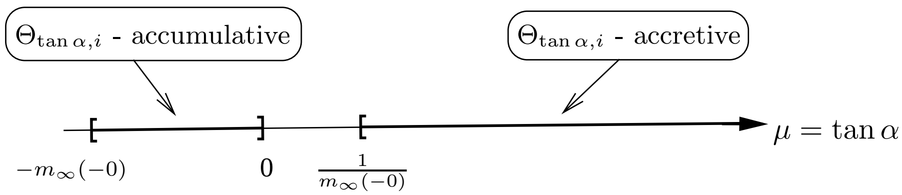

FIGURE 1. Accumulative and accretive L-systems $\Theta_{\tan\alpha,i}$.

Consider the functions $m_\alpha(z)$ described by (25)-(26) and associated with the non-negative operator $\dot{A}$ above. Let us observe how the parameter α in the definition of $m_\alpha(z)$ effects the L-system realizing $(-m_\alpha(z))$. Part of this question was answered in [**7**, Theorem 6.3]. It was shown that if the non-negative symmetric Schrödinger operator is such that $m_\infty(-0) \ge 0$, then the L-system $\Theta_{\tan\alpha,i}$ of the form (29) realizing the function $(-m_\alpha(z))$ is accretive if and only if

$$\tan\alpha \ge \frac{1}{m_\infty(-0)}. \tag{39}$$

We are going to use inequality (37) to see the values of $\mu = \tan\alpha$ that generate accumulative L-systems $\Theta_{\tan\alpha,i}$. This approach yields

$$-m_\infty(-0) \le \tan\alpha \le 0. \tag{40}$$

The established criteria for a function $(-m_\alpha(z))$ to be realized with an accretive or accumulative L-system $\Theta_{\tan\alpha,i}$ are graphically shown on Figure 1. This figure describes the dependence of the properties of realizing $(-m_\alpha(z))$ L-systems on the value of μ and hence α. The bold part of the real line depicts values of $\mu = \tan\alpha$ that produce accretive or accumulative L-systems $\Theta_{\mu,i}$.

Note that if $m_\infty(-0) = 0$ in (39), then $\alpha = \pi/2$ and $-m_{\frac{\pi}{2}}(z) = 1/m_\infty(z)$. Moreover, we know that if $m_\infty(-0) \ge 0$, then $1/m_\infty(z)$ is realized by an accretive system $\Theta_{\infty,i}$ (see [**7**, Theorem 6.2]). We also note that when $\tan\alpha = 0$ and hence $\alpha = 0$ we obtain $m_0(z) = m_\infty(z)$, and the realizing $-m_\infty(z)$ Schrödinger L-system is $\Theta_{0,i}$. The following theorem shows how the additional requirement of non-negativity affects the realization of functions $-m_\infty(z)$ and $1/m_\infty(z)$.

THEOREM 6. *Let $\dot{A}$ be a non-negative symmetric Schrödinger operator of the form* (15) *with deficiency indices* $(1,1)$ *and locally summable potential in* $\mathcal{H} = L^2[\ell,\infty)$. *If $m_\infty(z)$ is the Weyl-Titchmarsh function of $\dot{A}$ such that $m_\infty(-0) \ge 0$, then the L-system $\Theta_{0,i}$ realizing the function $(-m_\infty(z))$ is accumulative and the L-system $\Theta_{\infty,i}$ realizing the function $1/m_\infty(z)$ is accretive.*

PROOF. Since $m_\infty(-0) \ge 0$, we can apply (40) to conclude that $-m_0(z) = -m_\infty(z) \le 0$ implies that the L-system $\Theta_{0,i}$ realizing the function $(-m_\infty(z))$ is accumulative (see [**1**, Section 9.9]). The fact that the L-system $\Theta_{\infty,i}$ realizing the function $1/m_\infty(z)$ is accretive under the conditions of current theorem was proved in [**7**]. □

REMARK 7. *Some of analytic properties of the functions $(-m_\infty(z))$, $1/m_\infty(z)$, and $(-m_\alpha(z))$ were described in* [**7**, Theorem 6.5]. *Taking into account these results and the above reasoning we have that under the current set of assumptions:*

(1) the function $1/m_\infty(z)$ is Stieltjes if and only if $m_\infty(-0)\ge 0$;
(2) the function $(-m_\infty(z))$ is inverse Stieltjes if and only if $m_\infty(-0)\ge 0$;
(3) the function $(-m_\alpha(z))$ given by (26) is Stieltjes if and only if

$$0<\frac{1}{m_\infty(-0)}\le\tan\alpha,$$

and inverse Stieltjes if and only if

$$-m_\infty(-0)\le\tan\alpha\le 0.$$

Now once we established a criteria for an L-system realizing $(-m_\alpha(z))$ to be accumulative, we can look into more of its properties. We are going to turn to the case when our realizing L-system $\Theta_{\tan\alpha,i}$ is accumulative sectorial. To begin with let $\Theta_{\mu,h}$ be an L-system of the form (21), where $\mathbb{A}_{\mu,h}$ is an accumulative $(*)$-extension (17) of the accretive Schrödinger operator T_h. Let also $\tilde{\mathbb{A}}_{\mu,h}$ be of the form (38). Below is the list of some known facts about possible accumulativity and sectoriality of $\Theta_{\mu,h}$.

- If $\tilde{\mathbb{A}}_{\mu,h}$ of the form (38) is β-sectorial, then the impedance function $V_{\Theta_{\mu,h}}(z)$ defined by (6) belongs to the class S^{-1,β_1,β_2}.
- The operator T_h of $\Theta_{\mu,h}$ is $(\beta_2-\beta_1)$-sectorial with the exact angle of sectoriality $(\beta_2-\beta_1)$, and $\tan\beta_2\le\tan\beta$.
- In the case when $\beta_1=0$ and $\beta_2=\pi/2$ the operator T_h is accretive but not β-sectorial.
- If β is the exact angle of sectoriality of the operator T_h, then $V_{\Theta_{\mu,h}}(z)\in S^{-1,0,\beta}$.
- if the impedance function $V_{\Theta_{\mu,h}}(z)$ belongs to the class S^{-1,β_1,β_2}, then $\tilde{\mathbb{A}}_{\mu,h}$ is β-sectorial, where $\tan\beta$ is defined via (13).
- Both $\tilde{\mathbb{A}}_{\mu,h}$ and T_h are β-sectorial operators with the exact angle $\beta\in(0,\pi/2)$ if and only if $V_{\Theta_{\mu,h}}(z)\in S^{-1,0,\beta}$ and $\tan\beta$ is given by (14).

Consider a function $(-m_\alpha(z))$ and Schrödinger L-system $\Theta_{\tan\alpha,i}$ of the form (29) that realizes it. According to Theorem 6 this L-system $\Theta_{\tan\alpha,i}$ can be accumulative if and only if (40) holds, that is $-m_\infty(-0)\le\tan\alpha\le 0$. Moreover, according to [8, Theorem 6], $\Theta_{\tan\alpha,i}$ is accumulative sectorial if and only if

$$-m_\infty(-0)\le\tan\alpha<0, \tag{41}$$

and accumulative extremal (see [8, Theorem 7]) if and only if $\tan\alpha=0$. Also, if we assume that L-system $\Theta_{\tan\alpha,i}$ is β-sectorial, then its impedance function $V_{\Theta_{\tan\alpha,i}}(z)=-m_\alpha(z)$ belongs (see [12]) to certain sectorial classes of inverse Stieltjes functions discussed in Section 3. Namely, $(-m_\alpha(z))\in S^{-1,\beta}$. The following theorem provides more refined properties of $(-m_\alpha(z))$ for this case.

Theorem 8. *Let $\Theta_{\tan\alpha,i}$ be the accumulative L-system of the form (29) realizing the function $(-m_\alpha(z))$ associated with the non-negative operator $\dot{A}$. Let also $\tilde{\mathbb{A}}_{\tan\alpha,i}$ be a β-sectorial operator associated with $\Theta_{\tan\alpha,i}$ and defined by (7). Then the function $(-m_\alpha(z))$ belongs to the class S^{-1,β_1,β_2}, $\tan\beta_2\le\tan\beta$, and*

$$\tan\beta_1=\frac{\tan\alpha+m_\infty(-0)}{1-(\tan\alpha)m_\infty(-0)}, \tag{42}$$

and

$$\tan\beta_2=-\cot\alpha. \tag{43}$$

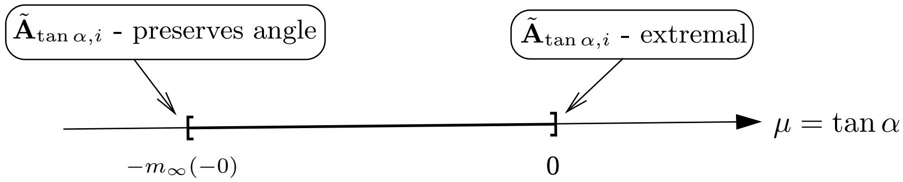

FIGURE 2. Associated operator $\tilde{\mathbb{A}}_{\tan\alpha,i}$.

Moreover, the operator T_i is $(\beta_2-\beta_1)$-sectorial with the exact angle of sectoriality $(\beta_2-\beta_1)$.

PROOF. It is given that $\Theta_{\tan\alpha,i}$ is accumulative and hence (41) holds. For further convenience we re-write $(-m_\alpha(z))$ as

$$-m_\alpha(z)=\frac{\sin\alpha+m_\infty(z)\cos\alpha}{-\cos\alpha+m_\infty(z)\sin\alpha}=\frac{\tan\alpha+m_\infty(z)}{(\tan\alpha)m_\infty(z)-1}. \tag{44}$$

Since under our assumption $\tilde{\mathbb{A}}_{\tan\alpha,i}$ is β-sectorial, then (see [**12**], [**8**]) the impedance function $V_{\Theta_{\tan\alpha,i}}(z)=-m_\alpha(z)$ belongs to certain sectorial classes discussed in Section 3. Particularly, $-m_\alpha(z)\in S^{-1,\beta}$ and $-m_\alpha(z)\in S^{-1,\beta_1,\beta_2}$, where (see [**8**])

$$\tan(\pi-\beta_1)=-\tan\beta_1=\lim_{x\to-0}(-m_\alpha(x))=\frac{\tan\alpha+m_\infty(-0)}{(\tan\alpha)m_\infty(-0)-1},$$

and

$$\begin{aligned}\tan(\pi-\beta_2)=-\tan\beta_2=\lim_{x\to-\infty}(-m_\alpha(x))&=\frac{\tan\alpha+m_\infty(-\infty)}{(\tan\alpha)m_\infty(-\infty)-1}\\&=\frac{\frac{\tan\alpha}{m_\infty(-\infty)}+1}{\tan\alpha-\frac{1}{m_\infty(-\infty)}}=\frac{1}{\tan\alpha}=\cot\alpha.\end{aligned}$$

Multiplying the above by (-1) one confirms (42) and (43). In order to show the rest, we apply [**12**, Theorem 9]. This theorem states that if $\tilde{\mathbb{A}}$ is a β-sectorial operator of the form (6) associated to an accumulative L-system Θ, then the impedance function $V_\Theta(z)$ belongs to the class S^{-1,β_1,β_2}, $\tan\beta_2\le\tan\beta$, and T is $(\beta_2-\beta_1)$-sectorial with the exact angle of sectoriality $(\beta_2-\beta_1)$. □

The next theorem explains two "endpoint" cases of accumulative realization for the function $(-m_\alpha(z))$.

THEOREM 9. *Let $\Theta_{\tan\alpha,i}$ be the accumulative L-system of the form (29) realizing the function $(-m_\alpha(z))$ with a sectorial main operator T_i whose exact angle of sectoriality is $\beta\in(0,\pi/2)$. Let also $\tilde{\mathbb{A}}_{\tan\alpha,i}$ be an associated operator defined by (6). Then*

(1) $\tilde{\mathbb{A}}_{\tan\alpha,i}$ is β-sectorial (with the same angle of sectoriality as T_i) if and only if $\tan\alpha=-m_\infty(-0)$ in (38);

(2) $\tilde{\mathbb{A}}_{\tan\alpha,i}$ is accretive but not β-sectorial for any $\beta\in(0,\pi/2)$ if and only if in (17) $\alpha=0$.

PROOF. The proof directly follows from [**8**, Theorems 6 and 7] after one sets $\mu=\tan\alpha=-m_\infty(-0)$ for part (1) and $\mu=\operatorname{Re}h=\tan 0=0$ for part (2). □

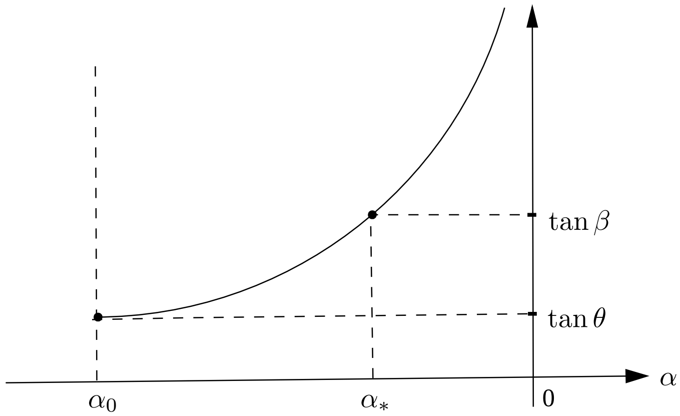

FIGURE 3. Angle of sectoriality β. Here $\alpha_0 = \arctan(-m_\infty(-0))$.

The result of Theorem 9 is graphically illustrated by Figure 2. Also we have shown that within the conditions of Theorem 9 the α-sectorial sesquilinear form (f, Tf) defined on a subspace $\operatorname{Dom}(T)$ of $\mathcal{H}_+$ can be extended to the α-sectorial form $(\tilde{\mathbb{A}}f, f)$ defined on $\mathcal{H}_+$ preserving the exact (for both forms) angle of sectoriality α. A general problem of extending sectorial sesquilinear forms was mentioned by T. Kato in [**19**].

Now we state and prove the following.

THEOREM 10. *Let $\Theta_{\tan\alpha,i}$ be an accumulative L-system of the form (29) that realizes $(-m_\alpha(z))$ with the main θ-sectorial operator T_i whose exact sectoriality angle is θ. Let also $\alpha_* \in (\arctan(-m_\infty(-0)), 0)$ be a fixed value that defines the associated operator $\tilde{\mathbb{A}}_{\tan\alpha_*,i}$ via (6), (17), and $(-m_\alpha(z)) \in S^{-1,\beta_1,\beta_2}$. Then the associated operator $\tilde{\mathbb{A}}_{\tan\alpha,i}$ is β-sectorial for any $\alpha \in (\arctan(-m_\infty(-0)), \alpha_*)$ with*

$$\tan\beta = \tan\beta_1 + 2\sqrt{\tan\beta_1\,\tan\beta_2}. \tag{45}$$

Moreover, if $\alpha = \arctan(-m_\infty(-0))$, then

$$\beta = \theta = \arctan\left(\frac{1}{m_\infty(-0)}\right).$$

PROOF. We note first that the conditions of our theorem imply the following: $\tan\alpha_* \in (-m_\infty(-0), 0)$. Thus, according to [**7**, Theorem 8] applied for $\mu = \tan\alpha$ the operator $\tilde{\mathbb{A}}_{\tan\alpha,i}$ is β-sectorial for some $\beta \in (0, \pi/2)$ for any α such that

$$-m_\infty(-0) \le \tan\alpha < \tan\alpha_*.$$

Formula (45) also follows from the corresponding formula in [**7**, Theorem 8] taken into account that β_1 and β_2 are defined via (42) and (43), respectively. Finally, since T_i is θ-sectorial, formula (34) yields $\tan\theta = \frac{1}{m_\infty(-0)}$. Applying part (1) of Theorem 9 gives us that $\beta = \theta$. This completes the proof. □

Note that Theorem 10 provides us with a value β which serves as a universal angle of sectoriality for the entire indexed family of associated operators $\tilde{\mathbb{A}}$ of the form (38) as depicted on Figure 3.

7. Example

Consider the differential expression with the Bessel potential

$$l_\nu = -\frac{d^2}{dx^2} + \frac{\nu^2 - 1/4}{x^2}, \quad x \in [1, \infty)$$

of order $\nu > 0$ in the Hilbert space $\mathcal{H} = L^2[1, \infty)$. The minimal symmetric operator

$$\begin{cases} \dot{A}\, y = -y'' + \frac{\nu^2 - 1/4}{x^2} y \\ y(1) = y'(1) = 0 \end{cases} \tag{46}$$

generated by this expression and boundary conditions has defect numbers $(1, 1)$. Let $\nu = 3/2$. It is known [1] that in this case

$$m_\infty(z) = 1 - \frac{iz}{\sqrt{z} + i}$$

and $m_\infty(-0) = 1$. The minimal symmetric operator then becomes

$$\begin{cases} \dot{A}\, y = -y'' + \frac{2}{x^2} y \\ y(1) = y'(1) = 0. \end{cases} \tag{47}$$

Consider operator T_h of the form (16) that is written for $h = i$ as

$$\begin{cases} T_i\, y = -y'' + \frac{2}{x^2} y \\ y'(1) = i\, y(1) \end{cases} . \tag{48}$$

This operator T_i will be shared as the main operator by the family of L-systems realizing functions $(-m_\alpha(z))$ in (25)-(26). It is accretive and β-sectorial since $\operatorname{Re} h = 0 > -m_\infty(-0) = -1$ and has the exact angle of sectoriality given by (see (34))

$$\tan \beta = \frac{\operatorname{Im} h}{\operatorname{Re} h + m_\infty(-0)} = \frac{1}{0+1} = 1 \quad \text{ or } \quad \beta = \frac{\pi}{4}. \tag{49}$$

The family of L-systems $\Theta_{\tan\alpha, i}$ of the form (29) that realizes functions

$$-m_\alpha(z) = \frac{(\sqrt{z} - iz + i)\cos\alpha + (\sqrt{z} + i)\sin\alpha}{(\sqrt{z} - iz + i)\sin\alpha - (\sqrt{z} + i)\cos\alpha}, \tag{50}$$

was constructed in [7]. According to (40) the L-systems $\Theta_{\tan\alpha, i}$ in (29) are accumulative if

$$-1 = -m_\infty(-0) \le \tan\alpha \le 0.$$

Applying part (2) of Theorem 9, we get that the realizing L-system $\Theta_{\tan\alpha, i}$ in (29) is such that the associated operator $\tilde{\mathbb{A}}_{\tan\alpha, i}$ is extremal accretive if $\mu = \tan\alpha = 0$ or $\alpha = 0$. Therefore the L-system

$$\Theta_{0,i} = \begin{pmatrix} \mathbb{A}_{0,i} & K_{0,i} & 1 \\ \mathcal{H}_+ \subset L_2[1, +0) \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix}, \tag{51}$$

where

$$\begin{aligned} \mathbb{A}_{0,i}\, y &= -y'' + \frac{2}{x^2} y - i\,[y'(1) - iy(1)]\,\delta'(x-1), \\ \mathbb{A}^*_{0,i}\, y &= -y'' + \frac{2}{x^2} y + i\,[y'(1) + iy(1)]\,\delta'(x-1), \end{aligned} \tag{52}$$

$K_{0,i}c = cg_{0,i}$, $(c \in \mathbb{C})$ and $g_{0,i} = \delta'(x-1)$. This L-system $\Theta_{0,i}$ realizes the function $-m_0(z) = -m_\infty(z)$. Also,

$$\begin{aligned} V_{\Theta_{0,i}}(z) &= -m_0(z) = -m_\infty(z) = \frac{iz}{\sqrt{z}+i} - 1 \\ W_{\Theta_{0,i}}(z) &= -\frac{m_\infty(z)-i}{m_\infty(z)+i} = \frac{(i-1)\sqrt{z}+iz-1-i}{(1+i)\sqrt{z}-iz-1+i}. \end{aligned} \tag{53}$$

The associate operator $\tilde{\mathbb{A}}_{0,i}$ is given by (38) as

$$\begin{aligned} \tilde{\mathbb{A}}_{0,i}\, y &= -y'' + \frac{2}{x^2}y - y'(1)\delta(x-1) - y(1)\delta'(x-1) + [y(1)+iy'(1)]\,\delta'(x-1) \\ &= -y'' + \frac{2}{x^2}y - y'(1)[\delta(x-1) - i\delta'(x-1)]. \end{aligned}$$

The adjoint operator $\tilde{\mathbb{A}}_{0,i}$ is

$$\tilde{\mathbb{A}}^*_{0,i} y = -y'' + \frac{2}{x^2}y - y'(1)[\delta(x-1) + i\delta'(x-1)],$$

and consequently

$$\mathrm{Re}\,\tilde{\mathbb{A}}_{0,i}\, y = -y'' + \frac{2}{x^2}y - y'(1)\delta(x-1) \quad \text{and} \quad \mathrm{Im}\,\tilde{\mathbb{A}}_{0,i}\, y = y'(1)\delta'(x-1).$$

The operator $\tilde{\mathbb{A}}_{0,i}$ above is accretive according to [**12**] which is also independently confirmed by direct evaluation

$$(\mathrm{Re}\,\tilde{\mathbb{A}}_{0,i}\, y, y) = \|y'(x)\|^2_{L^2} + 2\|y(x)/x\|^2_{L^2} \geq 0.$$

Moreover, according to Theorem 9 it is extremal, that is accretive but not β-sectorial for any $\beta \in (0, \pi/2)$. Indeed, it is easy to see that

$$(\mathrm{Im}\,\tilde{\mathbb{A}}_{0,i}\, y, y) = -|y'(1)|^2,$$

and hence we can have inequality (2) for all $y \in \mathcal{H}_+$ only if $\beta = \frac{\pi}{2}$. Thus, this is the case of the extremal operator. In addition, we have shown that the function $-m_0(z) = -m_\infty(z) = \frac{iz}{\sqrt{z}+i} - 1$ in (53) belongs to the sectorial class $S^{-1,0,\frac{\pi}{2}}$ of inverse Stieltjes functions.

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