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On Krein's formula in indefinite metric spaces

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Abstract

In this paper we extend some of the recent results in connection with the Krein resolvent formula which provides a complete description of all canonical resolvents and utilizes Weyl–Titchmarsh functions in the spaces with indefinite metrics. We show that coefficients in Krein's formula can be expressed in terms of analogues of the von Neumann parametrization formulas in the indefinite case. We consider properties of Weyl–Titchmarsh functions and show that two Weyl–Titchmarsh functions corresponding to π-self-adjoint extensions of a densely defined π-symmetric operator are connected via linear-fractional transformation with the coefficients presented in terms of von Neumann's parameters.

Keywords: Krein's formula; Pontryagin space; Indefinite metrics

1. Introduction

In [9] Gesztesy, Makarov, and one of the authors revisited Krein's formula associated with self-adjoint extensions of a densely defined symmetric operator. They showed that the coefficients in Krein's formula can be expressed in terms of the classical von Neumann parametrization formulas. The purpose of this note is to generalize and extend some recent results [4,9] to the case of the space with indefinite metrics with finite indefinite rank. All operators are considered in Pontryagin spaces

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Π_κ with an indefinite inner product and hence the notions of adjoint, symmetric, and unitary operators are replaced with π-adjoint, π-symmetric, and π-unitary operators, respectively (see definitions in Section 2). The concept of Weyl–Titchmarsh function in spaces Π_κ, so called Q-function, was introduced and studied by Krein and Langer [12,13]. A systematic study of Weyl–Titchmarsh functions defined in terms of spaces of boundary values in Hilbert and Krein spaces was conducted in [3,5,6,14,15]. In this paper we follow the definition and approach developed in [7–9] and extend it to the indefinite case. We show that in the case of a Hilbert space ($\kappa = 0$) our results completely match the formulas established earlier in [9].

We conclude our note with an example where the main space is Π_1 (i.e. having indefinite rank of one). All the components of our framework, including the coefficients of the Krein formula and Weyl–Titchmarsh functions, are explicitly derived.

Throughout the paper we follow the notation of [9].

2. Operators in Pontryagin spaces Π_κ

We start with the basic construction following some results from the theory of operators in Π_κ spaces [11–13]. Let Π_κ be a Pontryagin space [2,11], i.e., a Hilbert space $\mathscr{H}$ where along with the usual scalar product (x, y) there is an indefinite scalar product

$$[x, y] = (Jx, y), \tag{1}$$

where $J = P_+ - P_-$ is a bounded linear operator such that $J = J^*$, $J^2 = I$, and P_+ and P_- are complementary orthoprojections, $P_+ + P_- = I$. Putting $\Pi_\pm = P_\pm \Pi_\kappa$ we have

$$\Pi_\kappa = \Pi_+ \boxplus \Pi_-, \quad \dim \Pi_- = \kappa. \tag{2}$$

Here and below the direct orthogonal sum with respect to an indefinite scalar product (1) is denoted by $\boxplus$ and called *π-orthogonal sum*. Similarly, the *π-orthogonal complement* of a lineal L will be denoted by $L^{[\perp]}$. The positive definite (x, y) and indefinite $[x, y]$ scalar products are related by

$$(x, y) = [x_+, y_+] - [x_-, y_-],$$
$$[x, y] = (x_+, y_+) - (x_-, y_-),$$

where $x = x_+ + x_-$, $y = y_+ + y_-$, $x_+, y_+ \in \Pi_+$, and $x_-, y_- \in \Pi_-$.

The set of vectors $f \in L$ that are π-orthogonal to L, i.e. $f[\perp]L$ is called [11] the *isotropic part* of the linear manifold L. If the isotropic part of L has non-zero elements we say that the scalar product $[\cdot, \cdot]$ is *degenerate* [12] on L. Denote by L_+ (respectively, L_-, L_0) the set of all $x \in \Pi_\kappa$ for which $[x, x] > 0$ (respectively, $[x, x] < 0$, $[x, x] = 0$). The set L_+ (respectively L_-, L_0) is called *positive* (*negative*, *neutral*) part of L. Every subspace $\mathscr{L} \in \Pi_\kappa$ can be decomposed into a direct sum of π-orthogonal subspaces

$$\mathscr{L} = \mathscr{L}_+ \boxplus \mathscr{L}_0 \boxplus \mathscr{L}_-,$$

where $\mathscr{L}_+$, $\mathscr{L}_0$, and $\mathscr{L}_-$ are, respectively, positive, neutral, and negative subspaces, some of which may degenerate into null subspaces. For a subspace $\mathscr{L}$ above we write $\operatorname{sign}\mathscr{L} = (l_+, l_0, l_-)$ where $l_\pm = \dim \mathscr{L}_\pm$ and $l_0 = \dim \mathscr{L}_0$ [12].

Let $\dot{A}$ be a closed linear operator in Π_κ with a domain $\mathscr{D}(\dot{A})$ that is dense in Π_κ. The operator $\dot{A}^+$ is called *π-adjoint* to $\dot{A}$ if its domain $\mathscr{D}(\dot{A}^+)$ consists of elements $g \in \Pi_\kappa$ such that there exist $h \in \Pi_\kappa$ and

$$[\dot{A}f, g] = [f, h], \ \ \forall f \in \mathscr{D}(\dot{A}), \quad \text{and} \quad \dot{A}^+ g = h.$$

An operator $\dot{A}$ is said to be *π-symmetric* if $\dot{A} \subseteq \dot{A}^+$, i.e. $[\dot{A}f, g] = [f, \dot{A}g]$, for all $f \in \mathscr{D}(\dot{A})$, and *π-self-adjoint* if $\dot{A} = \dot{A}^+$.

It is easy to see that

$$\dot{A}^+ = J\dot{A}^* J,$$

where $\dot{A}^*$ is the operator in $\mathscr{H}$ adjoint to $\dot{A}$.

We recall [11] that a π-symmetric operator $\dot{A}$ in Π_κ cannot have more than κ eigenvalues, counting multiplicities, in the upper (lower) half-plane. If the operator A is π-self-adjoint, then these non-real eigenvalues are located symmetrically with respect to the real axis. For an arbitrary complex number λ and a π-symmetric operator $\dot{A}$ in Π_κ we set [11]

$$\mathscr{M}_\lambda = (\dot{A} - \lambda)\mathscr{D}(\dot{A}), \quad \mathscr{N}_{\bar{\lambda}} = \mathscr{M}_\lambda^{[\perp]}. \tag{3}$$

If λ ($\operatorname{Im}\lambda \neq 0$) is not an eigenvalue of $\dot{A}$, then $\mathscr{M}_\lambda$ is a subspace of Π_κ and $\mathscr{N}_\lambda$ is called [11] a *deficiency subspace* corresponding to λ. The number $n_+ = \dim \mathscr{N}_\lambda$ is called [11] an *upper deficiency index* of $\dot{A}$ in Π_κ and has the same value for all points λ with $\operatorname{Im}\lambda > 0$ that are not eigenvalues. Similarly we define a *lower deficiency index* $n_- = \dim \mathscr{N}_\lambda$ for all points λ with $\operatorname{Im}\lambda < 0$ that are not eigenvalues as well. The two values n_+ and n_- are, in general, different. Let $\Delta_{\dot{A}}$ be the set of all non-real λ for which the scalar product $[\cdot, \cdot]$ is degenerate on $\mathscr{N}_\lambda$. According to [12] the set $\Delta_{\dot{A}}$ of a π-symmetric operator $\dot{A}$ contains no interior points, its complement $(\mathbb{C}_+ \cup \mathbb{C}_-) \setminus \Delta_{\dot{A}}$ is an open set, and on every component of this open set $\operatorname{sign}\mathscr{N}_\lambda$ is constant.

It was shown in [12] that every π-symmetric operator $\dot{A}$ in the space Π_κ admits π-self-adjoint extensions in Π_κ if and only if its deficiency indices coincide. Also according to [12] for every π-symmetric operator $\dot{A}$ there is a number $\varrho_{\dot{A}} > 0$ such that the spectrum of every π-self-adjoint extension A of $\dot{A}$ lies in the strip $\{z \mid |\operatorname{Im} z| < \varrho_{\dot{A}}\}$. For the sake of simplicity we will only consider π-symmetric operators $\dot{A}$ with $\varrho_{\dot{A}} < 1$. As it also follows from the Krein–Langer theorem [12], the deficiency subspaces $\mathscr{N}_{\pm i}$ of such a π-symmetric operator $\dot{A}$ with equal deficiency indices are always positive.

3. Self-adjoint extensions in Π_κ

Let $\dot{A}$ be a closed densely defined symmetric operator in Π_κ with equal deficiency indices $\operatorname{def}(\dot{A}) = (n, n)$ and $\varrho_{\dot{A}} < 1$. We denote by $\mathscr{N}_\pm$ the deficiency subspaces of $\dot{A}$ corresponding to i, and note that

$$\mathscr{N}_\pm = \ker(\dot{A}^+ \mp \mathrm{i}).$$

For any π-self-adjoint extension A of $\dot{A}$ in Π_κ with $\zeta \in \rho(A)$, $\operatorname{Im}\zeta \neq 0$ its π-unitary Cayley transform $C_{A,\zeta}$ is given by

$$C_{A,\zeta} = \left(A - \bar{\zeta}\right)(A - \zeta)^{-1}.$$

Let A be a π-self-adjoint extension of $\dot{A}$ in Π_κ. Since $\varrho_{\dot{A}} < 1$ we get that $\mathrm{i} \in \rho(A)$, the resolvent set of A. We introduce

$$C_A = (A + \mathrm{i})(A - \mathrm{i})^{-1}. \tag{4}$$

In addition, we remind that two self-adjoint extensions A_1 and A_2 of $\dot{A}$ *relatively prime* if $\mathscr{D}(A_1) \cap \mathscr{D}(A_2) = \mathscr{D}(\dot{A})$. The direct sum of two linear subspaces $\mathscr{V}$ and $\mathscr{W}$ of $\mathscr{H}$ is denoted by $\mathscr{V} \dotplus \mathscr{W}$ in the following.

The following lemma is a modification of the similar result in [4,9] for the case of spaces Π_κ.

Lemma 1. *Let A, A_1, and A_2 be π-self-adjoint extensions of $\dot{A}$ such that* i *is not an eigenvalue for either operator. Then*

(i) *The Cayley transform of A maps $\mathscr{N}_-$ onto $\mathscr{N}_+$*

$$C_A \mathscr{N}_- = \mathscr{N}_+. \tag{5}$$

(ii) $\mathscr{D}(A) = \mathscr{D}(\dot{A}) \dotplus (I - C_A^{-1})\mathscr{N}_+$.

(iii) *$\mathscr{N}_+$ is an invariant subspace for $C_{A_1} C_{A_2}^{-1}$ and $C_{A_2} C_{A_1}^{-1}$. In addition, A_1 and A_2 are relatively prime if and only if*

$$1 \notin \sigma_p\left(C_{A_1} C_{A_2}^{-1}\Big|_{\mathscr{N}_+} \right). \tag{6}$$

(iv) *Suppose A_1 and A_2 are relatively prime w.r.t. $\dot{A}$. Then*

$$\overline{\operatorname{ran}\left((A_2 - \mathrm{i})^{-1} - (A_1 - \mathrm{i})^{-1}\right)} = \mathscr{N}_+, \tag{7}$$

$$\ker\left(\left((A_2 - \mathrm{i})^{-1} - (A_1 - \mathrm{i})^{-1}\right)\Big|_{\mathscr{N}_-} \right) = \{0\}. \tag{8}$$

Proof. Most of these facts are standard and their proofs can be replicated from the proof of the relevant lemma in [9] with some minor adjustments due to the indefinite metrics of the space Π_κ. That is why we only sketch the main steps.

(i) Pick $g \in \mathscr{D}(\dot{A})$, $f = (\dot{A} - \mathrm{i})g$, then $C_A f = (\dot{A} + \mathrm{i})g \in \mathrm{ran}(\dot{A} + \mathrm{i})$ yields $C_A \mathrm{ran}(\dot{A} - \mathrm{i}) \subseteq \mathrm{ran}(\dot{A} + \mathrm{i})$. Similarly one infers $C_A^{-1}\mathrm{ran}(\dot{A} + \mathrm{i}) \subseteq \mathrm{ran}(\dot{A} - \mathrm{i})$ and hence $C_A \mathrm{ran}(\dot{A} - \mathrm{i}) = \mathrm{ran}(\dot{A} + \mathrm{i})$. Since C_A is π-unitary, $C_A \overline{\mathrm{ran}(\dot{A} - \mathrm{i})} = \overline{\mathrm{ran}(\dot{A} + \mathrm{i})}$. Positive definiteness of $\mathscr{N}_+$ [12] and $\mathscr{H} = \ker(\dot{A}^+ - \mathrm{i}) \boxplus \overline{\mathrm{ran}(\dot{A} + \mathrm{i})}$ then yield $C_A \mathscr{N}_- = \mathscr{N}_+$.
(ii) By the analogues of von Neumann's formula for the case of densely defined operator in Π_κ [11], we have

$$\mathscr{D}(A) = \mathscr{D}(\dot{A}) \dotplus \mathscr{N}_+ \dotplus \mathscr{U}_A \mathscr{N}_+ \tag{9}$$

for some linear π-isomorphism $\mathscr{U}_A : \mathscr{N}_+ \to \mathscr{N}_-$. Since $I - C_A^{-1} = 2\mathrm{i}(A + \mathrm{i})^{-1}$, $(I - C_A^{-1})\mathscr{N}_+ = 2\mathrm{i}(A + \mathrm{i})^{-1}\mathscr{N}_+ \subseteq \mathscr{D}(A)$, one concludes

$$\mathscr{U}_A = -C_A^{-1}\Big|_{\mathscr{N}_+}. \tag{10}$$

(iv) Let $g \in \mathscr{D}(\dot{A})$, $f = (\dot{A} + \mathrm{i})g$, then for all $h \in \mathscr{H}$

$$\begin{aligned}&\left[f, \left((A_2 - \mathrm{i})^{-1} - (A_1 - \mathrm{i})^{-1}\right) h\right] \\ &\quad = \left[\left((A_2 + \mathrm{i})^{-1} - (A_1 + \mathrm{i})^{-1}\right)(\dot{A} + \mathrm{i})g, h\right] = 0,\end{aligned}$$

yields

$$\mathrm{ran}((A_2 - \mathrm{i})^{-1} - (A_1 - \mathrm{i})^{-1}) \subseteq \mathrm{ran}(\dot{A} + \mathrm{i})^{[\perp]} = \ker(\dot{A}^+ - \mathrm{i}) = \mathscr{N}_+.$$

Next, let $0 \neq f_+ \in \mathscr{N}_+$ and $f_+[\perp]\mathrm{ran}\left((A_2 - \mathrm{i})^{-1} - (A_1 - \mathrm{i})^{-1}\right)$. In particular,

$$f_+[\perp]\left((A_2 - \mathrm{i})^{-1} - (A_1 - \mathrm{i})^{-1}\right) C_{A_1}^{-1} f_+.$$

Using that, $(A_1 - \mathrm{i})^{-1}C_{A_1}^{-1} f_+ = -(\mathrm{i}/2)\left(I - C_{A_1}^{-1}\right) f_+$ and

$$\begin{aligned}(A_2 - \mathrm{i})^{-1}C_{A_1}^{-1} f_+ &= (A_2 - \mathrm{i})^{-1}C_{A_2}^{-1}\left(C_{A_2}C_{A_1}^{-1} f_+\right) \\ &= -(\mathrm{i}/2)\left(I - C_{A_2}^{-1}\right)\left(C_{A_2}C_{A_1}^{-1} f_+\right) \\ &= -(\mathrm{i}/2)\left(C_{A_2}C_{A_1}^{-1} - C_{A_1}^{-1}\right) f_+,\end{aligned}$$

and hence

$$\left((A_2 - \mathrm{i})^{-1} - (A_1 - \mathrm{i})^{-1}\right) C_{A_1}^{-1} f_+ = -(\mathrm{i}/2)\left(C_{A_2}C_{A_1}^{-1} - I\right) f_+. \tag{11}$$

Thus, $f_+[\perp]\left(C_{A_2}C_{A_1}^{-1} - I\right) f_+$, that is,

$$\left[f_+, C_{A_2}C_{A_1}^{-1} f_+\right] = [f_+, f_+].$$

Since $C_{A_2}C_{A_1}^{-1}\Big|_{\mathscr{N}_+}$ is π-unitary, one concludes $C_{A_2}C_{A_1}^{-1}f_+ = f_+ = C_{A_1}C_{A_2}^{-1}f_+$ and hence

$$1 \in \sigma_p\left(C_{A_1}C_{A_2}^{-1}|\mathscr{N}_+\right). \tag{12}$$

But (12) contradicts the hypothesis that A_1 and A_2 are relatively prime w.r.t. $\dot{A}$. Consequently, $\overline{\operatorname{ran}\left((A_2-\mathrm{i})^{-1} - \operatorname{ran}((A_1-\mathrm{i})^{-1})\right)} = \mathscr{N}_+$, which is (7).

To prove (8) we note that every $f_- \in \mathscr{N}_-$ is of the form $f_- = C_{A_1}^{-1}f_+$ for some $f_+ \in \mathscr{N}_+$ using (i). Suppose $((A_2-\mathrm{i})^{-1} - (A_1-\mathrm{i})^{-1})C_{A_1}^{-1}f_+ = 0$. By (11), this yields $C_{A_1}C_{A_2}^{-1}f_+ = f_+$ and hence $1 \in \sigma_p\left(C_{A_1}C_{A_2}^{-1}\Big|_{\mathscr{N}_+}\right)$. Since A_1 and A_2 are relatively prime w.r.t. $\dot{A}$ one concludes $f_- = C_{A_1}^{-1}f_+ = 0$. □

4. Function $P_{1,2}(z)$

Next, assuming A_ℓ, $\ell = 1, 2$ to be π-self-adjoint extensions of $\dot{A}$ and following [9], we define

$$\begin{aligned} P_{1,2}(z) = {} & (A_1 - z)(A_1 - \mathrm{i})^{-1}\left((A_2 - z)^{-1} - (A_1 - z)^{-1}\right) \\ & \times (A_1 - z)(A_1 + \mathrm{i})^{-1}, \quad z, \mathrm{i} \in \rho(A_1) \cap \rho(A_2). \end{aligned} \tag{13}$$

We collect the following properties of $P_{1,2}(z)$.

Lemma 2. *Let $z, z', \mathrm{i} \in \rho(A_1) \cap \rho(A_2)$.*

(i) $P_{1,2} : \rho(A_1) \cap \rho(A_2) \to [\Pi_\kappa, \Pi_\kappa]$ *is analytic and*

$$P_{1,2}(z)^+ = P_{1,2}(\bar{z}). \tag{14}$$

(ii)

$$P_{1,2}(z)\Big|_{\mathscr{N}_+^{[\perp]}} = 0, \quad P_{1,2}(z)\mathscr{N}_+ \subseteq \mathscr{N}_+. \tag{15}$$

(iii)

$$\begin{aligned} P_{1,2}(z) = {} & P_{1,2}(z') + (z - z')P_{1,2}(z')(A_1 + \mathrm{i}) \\ & \times (A_1 - z')^{-1}(A_1 - \mathrm{i})(A_1 - z)^{-1}P_{1,2}(z). \end{aligned} \tag{16}$$

(iv) $\operatorname{ran}\left(P_{1,2}(z)\,|_{\mathscr{N}_+}\right)$ *is independent of $z \in \rho(A_1) \cap \rho(A_2)$.*

(v) *Assume A_1 and A_2 are relatively prime π-self-adjoint extensions of $\dot{A}$. Then $P_{1,2}(z)|_{\mathscr{N}_+} : \mathscr{N}_+ \to \mathscr{N}_+$ is invertible (i.e., one-to-one).*

(vi) *Assume A_1 and A_2 are relatively prime π-self-adjoint extensions of $\dot{A}$. Then $\overline{\operatorname{ran}(P_{1,2}(\mathrm{i}))} = \mathscr{N}_+$.*

(vii)

$$P_{1,2}(\mathrm{i})\,\big|_{\mathcal{N}_+} = (\mathrm{i}/2)\left(I - C_{A_2}C_{A_1}^{-1}\right)\Big|_{\mathcal{N}_+}. \tag{17}$$

Next, let

$$C_{A_2}C_{A_1}^{-1}\Big|_{\mathcal{N}_+} = -\mathrm{e}^{-2\mathrm{i}\alpha_{1,2}} \tag{18}$$

for some π-self-adjoint (possibly unbounded) operator $\alpha_{1,2}$ in $\mathcal{N}_+$. If A_1 and A_2 are relatively prime, then

$$\left\{\left(m+\frac{1}{2}\right)\pi\right\}_{m\in\mathbb{Z}} \cap \sigma_p(\alpha_{1,2}) = \emptyset$$

and

$$\left(P_{1,2}(\mathrm{i})\big|_{\mathcal{N}_+}\right)^{-1} = \tan(\alpha_{1,2}) - \mathrm{i}I_{\mathcal{N}_+}. \tag{19}$$

In addition, $\tan(\alpha_{1,2}) \in [\mathcal{N}_+, \mathcal{N}_+]$ *if and only if* $\operatorname{ran}(P_{1,2}(\mathrm{i})) = \mathcal{N}_+$.

Proof. Most of the steps in the proof can be replicated from the corresponding result in [9] either directly or with some minor adjustments. We sketch the rest of the proof.
(i) is clear from (13) (see also [12]).
(ii) Let $f \in \mathscr{D}(\dot{A})$, $g = (\dot{A} + \mathrm{i})f$. Then

$$P_{1,2}(z)g = (A_1 - z)(A_1 - \mathrm{i})^{-1}\left((A_2 - z)^{-1} - (A_1 - z)^{-1}\right)(\dot{A} - z)f = 0$$

yields $P_{1,2}(z)|_{\operatorname{ran}(\dot{A}+\mathrm{i})} = 0$ and hence $P_{1,2}(z)|_{\overline{\operatorname{ran}(\dot{A}+\mathrm{i})}} = P_{1,2}(z)\Big|_{\mathcal{N}_+^{[\perp]}} = 0$ since $P_{1,2}(z) \in [\Pi_\kappa, \Pi_\kappa]$. Moreover, by (13)

$$\operatorname{ran}(P_{1,2}(z)) \subseteq (A_1 - z)(A_1 - \mathrm{i})^{-1}\ker(\dot{A}^+ - z) \subseteq \ker(\dot{A}^+ - \mathrm{i}) = \mathcal{N}_+$$

since

$$\begin{aligned}
&\left(\dot{A}^+ - \mathrm{i}\right)(A_1 - z)(A_1 - \mathrm{i})^{-1}\Big|_{\ker(\dot{A}^+ - z)} \\
&\quad = \left(\dot{A}^+ - \mathrm{i}\right)\left(I - (z - \mathrm{i})(A_1 - \mathrm{i})^{-1}\right)\Big|_{\ker(\dot{A}^+ - z)} \\
&\quad = \left((z - \mathrm{i})I - (z - \mathrm{i})\left(\dot{A}^+ - \mathrm{i}\right)(A_1 - \mathrm{i})^{-1}\right)\Big|_{\ker(\dot{A}^+ - z)} \\
&\quad = 0.
\end{aligned}$$

This proves (15).
(iii)–(vi) Proved in [9].
(vii) As we have already mentioned above the subspace $\mathcal{N}_+$ is positively definite, and thus all the results concerning the restriction onto $\mathcal{N}_+$ directly follow from [9] due to the fact that $\mathcal{N}_+$ is a Hilbert space with respect to $[\cdot, \cdot]$. □

5. Weyl–Titchmarsh operator and class N_κ

In this section we define the Weyl–Titchmarsh operator-functions associated with π-self-adjoint extensions of $\dot{A}$ and study their analytical properties.

Definition 3. Let A be a π-self-adjoint extension of $\dot{A}$, $\mathcal{N} \subseteq \mathcal{N}_+$ a closed linear subspace of $\mathcal{N}_+$, and $z \in \rho(A)$. Then the Weyl–Titchmarsh operator $M_{A,\mathcal{N}}(z) \in [\mathcal{N}, \mathcal{N}]$ associated with the pair $(A, \mathcal{N})$ is defined by

$$\begin{aligned} M_{A,\mathcal{N}}(z) &= P_{\mathcal{N}}(zA + I)(A - z)^{-1} P_{\mathcal{N}}\Big|_{\mathcal{N}} \\ &= zI_{\mathcal{N}} + \left(1 + z^2\right) P_{\mathcal{N}}(A - z)^{-1} P_{\mathcal{N}}\Big|_{\mathcal{N}} \end{aligned} \tag{20}$$

with $P_{\mathcal{N}}$ the π-orthogonal projection in $\mathcal{H}$ onto $\mathcal{N}$.

Let $\mathcal{N}$ be a Hilbert space with an inner product $(\cdot, \cdot)$ and an operator-valued function $Q(z)$ belongs to $[\mathcal{N}, \mathcal{N}]$.

Definition 4 [13]. We say that an operator-valued function $Q(z) \in [\mathcal{N}, \mathcal{N}]$ belongs to the class N_κ if it is meromorphic in the upper half-plane and the kernel

$$N_Q(z, \zeta) = \frac{Q(z) - Q^*(z)}{z - \bar{\zeta}} \tag{21}$$

has κ negative squares, i.e. the form

$$\sum_{j,k=0}^{n} (N_Q(z_j, z_k)h_j, h_k)\xi_j\bar{\xi}_k,$$
$$\forall z_j \in \mathbb{C}_+,\ h_j \in \mathcal{N},\ \xi_j \in \mathbb{C},\ j = 0, 1, \ldots, n, \tag{22}$$

contains no more than κ negative squares and for one such a set exactly κ negative squares.

In what follows we denote $\mathrm{Re}(T) = (T + T^+)/2$, $\mathrm{Im}(T) = (T - T^+)/2\mathrm{i}$ for linear operators T in Π_κ with $\mathcal{D}(T) = \mathcal{D}(T^+)$. Similarly, for a linear operator Q with $\mathcal{D}(Q) = \mathcal{D}(Q^*)$ in a Hilbert space we use the same notation to denote $\mathrm{Re}(Q) = (Q + Q^*)/2$ and $\mathrm{Im}(Q) = (Q - Q^*)/2\mathrm{i}$.

We note that since $\mathcal{N}_+$ is positively definite and $\mathcal{N}$ is a closed subspace of $\mathcal{N}_+$, one can consider $\mathcal{N}$ a Hilbert space with the scalar product $[x, y]$ for all $x, y \in \mathcal{N}$.

Theorem 5. *Let A be a π-self-adjoint extension of $\dot{A}$, $\mathcal{N}$ a closed subspace of $\mathcal{N}_+$. Then the Weyl–Titchmarsh operator $M_{A,\mathcal{N}}(z)$ belongs to the class $N_{\kappa'}$, $0 \leqslant \kappa' \leqslant \kappa$ and the following properties hold*:

(1) *$M_{A,\mathcal{N}}(z)$ is analytic in $z \in \mathbb{C}\backslash(\mathbb{R} \cup \sigma_p(A))$ and*
$$M_{A,\mathcal{N}}(\bar{z}) = M^*_{A,\mathcal{N}}(z). \tag{23}$$

(2) *For* $z \in \rho(A) \setminus \Delta_{\dot{A}}$, $|z| > \varrho_{\dot{A}}$

$$\operatorname{Im}(z) \operatorname{Im}(M_{A,\mathcal{N}}(z)) \geqslant 0. \tag{24}$$

(3) *For all* $f \in \mathcal{N}$

$$w - \lim_{y\uparrow\infty} \frac{M_{A,\mathcal{N}}(\mathrm{i}y)}{y} = \lim_{y\uparrow\infty} \frac{\big(M_{A,\mathcal{N}}(\mathrm{i}y) f, f\big)}{y} = 0. \tag{25}$$

(4) *For all* $f \in \mathcal{N}$, $f \neq 0$

$$\lim_{y\uparrow\infty} y \left(\operatorname{Im} M_{A,\mathcal{N}}(\mathrm{i}y) f, f\right) = \infty. \tag{26}$$

(5) $M_{A,\mathcal{N}}(z)$ *is normalized*, *that is*

$$M_{A,\mathcal{N}}(\mathrm{i}) = \mathrm{i} I_{\mathcal{N}}. \tag{27}$$

Proof. Even though some parts of the proof are parallel to the proof of a theorem in [13], we outline main steps for the convenience of the reader.
Using (20), an explicit computation yields

$$\frac{M_{A,\mathcal{N}}(z) - M^*_{A,\mathcal{N}}(\zeta)}{z - \bar{\zeta}} = I_{\mathcal{N}} + P_{\mathcal{N}} \frac{q(z)(A - z)^{-1} - q(\bar{\zeta})(A - \bar{\zeta})^{-1}}{z - \bar{\zeta}} P_{\mathcal{N}},$$

where $q(z) = z^2 + 1$ and $z, \zeta \in \mathbb{C}\backslash(\mathbb{R} \cup \sigma_p(A))$, $z \neq \bar{\zeta}$. Let

$$U_{\mathrm{i}z} = I + (z - \mathrm{i})(A - z)^{-1} = (A - \mathrm{i})(A - z)^{-1}. \tag{28}$$

Then by direct calculations one gets

$$\frac{M_{A,\mathcal{N}}(z) - M^*_{A,\mathcal{N}}(\zeta)}{z - \bar{\zeta}} = P_{\mathcal{N}} U^+_{\mathrm{i}\zeta} U_{\mathrm{i}z} P_{\mathcal{N}},$$
$$z, \zeta \in \mathbb{C}\backslash(\mathbb{R} \cup \sigma_p(A)),\ z \neq \bar{\zeta}. \tag{29}$$

Taking this into account (22) yields for $z_j \in \mathbb{C}_+\backslash\sigma_p(A)$, $h_j \in \mathcal{N}$, $\xi_j \in \mathbb{C}$, $j = 0, 1, \ldots, n$

$$\sum_{j,k=0}^{n} (M_{A,\mathcal{N}}(z_j, z_k) h_j, h_k) \xi_j \bar{\xi}_k = \sum_{j,k=0}^{n} [U_{\mathrm{i}z_j} h_j, U_{\mathrm{i}z_k} h_k] \xi_j \bar{\xi}_k. \tag{30}$$

Obviously, the right hand side of (30) has no more than κ negative squares. From the definition of $M_{A,\mathcal{N}}(z)$ (20) one can see that $M_{A,\mathcal{N}}(\bar{z}) = M^*_{A,\mathcal{N}}(z)$. In order to show (24) we will consider (29) for $z = \zeta$. The analyticity of $M_{A,\mathcal{N}}(z)$ on $\mathbb{C}\backslash(\mathbb{R} \cup \sigma_p(A))$ easily follows from (29) as well.

In order to prove (25) we follow [13] and rewrite $M_{A,\mathcal{N}}(z)$ in the form

$$M_{A,\mathcal{N}}(z) = -\mathrm{i} I_{\mathcal{N}} + (z + \mathrm{i}) P_{\mathcal{N}} (A - \mathrm{i})(A - z)^{-1} P_{\mathcal{N}}.$$

Then

$$\frac{M_{A,\mathcal{N}}(\mathrm{i}y)}{y} = -\frac{\mathrm{i}}{y} I_{\mathcal{N}} + \frac{y + \mathrm{i}}{y} P_{\mathcal{N}} (A - \mathrm{i})(A - \mathrm{i}y)^{-1} P_{\mathcal{N}},$$

and (25) becomes equivalent to

$$w - \lim_{y\uparrow\infty} P_{\mathscr{N}}(A-\mathrm{i})(A-\mathrm{i}y)^{-1}P_{\mathscr{N}} = 0.$$

As it was shown in [13], for any π-self-adjoint operator A in Π_κ the following decomposition holds

$$\Pi_\kappa = {\Pi_\kappa}' \boxplus \Pi_0, \tag{31}$$

where ${\Pi_\kappa}' \subseteq \mathscr{D}(A)$ is an invariant with respect to A subspace, P' and $P_0 = I - P'$ are π-orthogonal projection operators. Moreover,

$$A' = A\,\Big|_{{\Pi_\kappa}'}, \tag{32}$$

is a bounded π-self-adjoint in ${\Pi_\kappa}'$, Π_0 is a Hilbert space with the inner product $(f, g) = [f, g]$, $(\forall f, g \in \Pi_0)$, and

$$A_0 = A\,\Big|_{\Pi_0} \tag{33}$$

is a self-adjoint operator in Π_0. Then

$$P_{\mathscr{N}}(A-\mathrm{i})(A-\mathrm{i}y)^{-1}P_{\mathscr{N}} = P_{\mathscr{N}}(A'-\mathrm{i})(A'-\mathrm{i}y)^{-1}P_{\mathscr{N}} + P_{\mathscr{N}}(A_0-\mathrm{i})(A_0-\mathrm{i}y)^{-1}P_{\mathscr{N}}. \tag{34}$$

Since the operator A' is bounded, the first term in (34) behaves like $O\left(\frac{1}{y}\right)$ as $y \uparrow \infty$ and we should focus on the second term only. Consider the function

$$F(y; f, g) = \Big[P_{\mathscr{N}}(A_0-\mathrm{i})(A_0-\mathrm{i}y)^{-1}P_{\mathscr{N}}P_0 f, g\Big], \quad f, g \in \mathscr{N}. \tag{35}$$

For the self-adjoint operator A_0 in a Hilbert space Π_0 we have

$$(A_0 - z)^{-1} = \int_{-\infty}^{\infty} \frac{\mathrm{d}E_\lambda}{\lambda - z},$$

where E_λ is a spectral function of A_0. Then the function $F(y; f, g)$ takes a form

$$F(y; f, g) = \int_{-\infty}^{\infty} \frac{\lambda - \mathrm{i}}{\lambda - \mathrm{i}y}\,\mathrm{d}\sigma_{fg}(\lambda),$$

where $\sigma_{fg}(\lambda) = [E_\lambda P_0 f, P_0 g]$ is a function of bounded variation. One can see that there exists a constant $\gamma > 0$ such that for $y \geqslant \gamma$ we have

$$\left|\frac{\lambda - \mathrm{i}}{\lambda - \mathrm{i}y}\right| < 1.$$

On the other hand there is a constant $\beta > 0$ such that for $-\beta \leqslant \lambda \leqslant \beta$

$$\lim_{y\uparrow\infty}\left|\frac{\lambda - \mathrm{i}}{\lambda - \mathrm{i}y}\right| = 0.$$

Consequently,

$$\lim_{y\uparrow\infty} F(y;\, f, g) = 0,$$

and this completes the proof of (25).

In order to prove (26) we will show that

$$\lim_{y\uparrow\infty} y(\operatorname{Im} M_{A,\mathscr{N}}(\mathrm{i}y)f, f) < \infty, \quad f \in \mathscr{N}. \tag{36}$$

implies $f[\perp]\mathscr{D}(\dot{A})$ which contradicts that $\dot{A}$ is densely defined in Π_κ. We use decomposition (31) and (28), (32), and (33) to get

$$[U_{\mathrm{i}z}f, U_{\mathrm{i}z}f] = \left[U^0_{\mathrm{i}z}f, U^0_{\mathrm{i}z}f\right] + \left[U'_{\mathrm{i}z}f, U'_{\mathrm{i}z}f\right], \tag{37}$$

where $U^0_{\mathrm{i}z} = (A_0 - \mathrm{i})(A_0 - z)^{-1}$ and $U'_{\mathrm{i}z} = (A' - \mathrm{i})(A' - z)^{-1}$. Since the operator A' is bounded the second term in (37) behaves like $O\left(\frac{1}{z}\right)$ as $|z| \to \infty$. Using (29) for $z = \zeta = \mathrm{i}y$ we get

$$y(\operatorname{Im} M_{A,\mathscr{N}}(\mathrm{i}y)f, f) = y\int_{-\infty}^{\infty} \frac{|\lambda - \mathrm{i}|^2}{\lambda^2 + y^2}\, \mathrm{d}\sigma_f(\lambda) + O\left(\frac{1}{y}\right), \quad y \uparrow \infty, \tag{38}$$

where $\sigma_f(\lambda) = [E_\lambda P_0 f, P_0 f]$. One can easily see that (36) is equivalent to (38) which is also equivalent to

$$\int_{-\infty}^{\infty} \lambda^2 \mathrm{d}\sigma_f(\lambda) < \infty.$$

This last condition implies $P_0 f \in \mathscr{D}(A_0)$ and therefore $f \in \mathscr{D}(A)$. Then $f \in \mathscr{D}(A) \cap \mathscr{N} \subseteq \mathscr{N}_+$ which is possible only if $f = 0$ since $\overline{\mathscr{D}(\dot{A})} = \mathscr{H}$. Therefore (26) takes place.

The formula (27) can be proved by the direct substitution. □

Remark 6. We should note that the properties (25) and (26) were considered in a different environment in [1,13].

Remark 7. The result of the Theorem 5 is valid for $\kappa' = \kappa$ if the extension A is such that

$$\mathtt{cls}\left\{(A - z)^{-1}\mathscr{N},\; z \in \rho(A)\right\} = \Pi_\kappa. \tag{39}$$

Conversely, an operator-function of the class N_κ with properties (1)–(5) can be realized as a Weyl–Titchmarsh function $M_{A,\mathscr{N}}(z)$ of the form (20) associated with a π-self-adjoint extension A of some π-symmetric operator $\dot{A}$ with the property (39) (see [13]).

6. The Krein formula

We need the following lemma and theorem that are modified versions of the corresponding results from [9].

Lemma 8. *Let A_ℓ, $\ell = 1, 2$ be relatively prime π-self-adjoint extensions of $\dot{A}$. Then*

$$\begin{aligned}\left(P_{1,2}(z)\,\big|_{\mathscr{N}_+}\right)^{-1} &= \left(P_{1,2}(\mathrm{i})\,\big|_{\mathscr{N}_+}\right)^{-1} - (z-\mathrm{i})P_{\mathscr{N}_+}\\ &\quad\times (A_1+\mathrm{i})(A_1-z)^{-1}P_{\mathscr{N}_+}\\ &= \tan(\alpha_{1,2}) - M_{A_1,\mathscr{N}_+}(z), \quad z\in\rho(A_1).\end{aligned} \tag{40}$$

The theorem below presents Krein's resolvent formula in terms of Weyl–Titchmarsh operator-function in spaces Π_κ.

Theorem 9. *Let A_1 and A_2 be π-self-adjoint extensions of $\dot{A}$ and $z \in \rho(A_1) \cap \rho(A_2)$. Then*

$$\begin{aligned}(A_2-z)^{-1} &= (A_1-z)^{-1} + (A_1-\mathrm{i})(A_1-z)^{-1}P_{1,2}(z)\\ &\quad\times (A_1+\mathrm{i})(A_1-z)^{-1}\end{aligned} \tag{41}$$

$$\begin{aligned}&= (A_1-z)^{-1} + (A_1-\mathrm{i})(A_1-z)^{-1}P_{\mathscr{N}_{1,2,+}}\\ &\quad\times \left(\tan(\alpha_{\mathscr{N}_{1,2,+}}) - M_{A_1,\mathscr{N}_{1,2,+}}(z)\right)^{-1}P_{\mathscr{N}_{1,2,+}}\\ &\quad\times (A_1+\mathrm{i})(A_1-z)^{-1},\end{aligned} \tag{42}$$

where

$$\mathscr{N}_{1,2,+} = \ker\left(\left(A_1\,\big|_{\mathscr{D}(A_1)\cap\mathscr{D}(A_2)}\right)^+ - \mathrm{i}\right) \tag{43}$$

and

$$\mathrm{e}^{-2\mathrm{i}\alpha_{\mathscr{N}_{1,2,+}}} = -C_{A_2}C_{A_1}^{-1}\,\big|_{\mathscr{N}_{1,2,+}}. \tag{44}$$

Proof. If A_1 and A_2 are relatively prime w.r.t. $\dot{A}$, Lemmas 1, 2, and 8 prove (41)–(44). If A_1 and A_2 are arbitrary π-self-adjoint extensions of $\dot{A}$ one replaces $\dot{A}$ by the largest common symmetric part of A_1 and A_2 given by $A_1\,\big|_{\mathscr{D}(A_1)\cap\mathscr{D}(A_2)}$. □

Corollary 10

$$P_{1,2}(\mathrm{i})\,\big|_{\mathscr{N}_{1,2,+}} = (\mathrm{i}/2)\left(I - \mathscr{U}_{A_2}^{-1}\mathscr{U}_{A_1}\right)\big|_{\mathscr{N}_{1,2,+}}, \tag{45}$$

where

$$\mathscr{U}_{A_\ell} = -C_{A_\ell}^{-1}\,\big|_{\mathscr{N}_+}, \quad \ell = 1, 2 \tag{46}$$

denotes the linear π-isometric isomorphism from $\mathscr{N}_+$ onto $\mathscr{N}_-$ parameterizing the π-self-adjoint extensions A_ℓ of $\dot{A}$.

Proof. Combine (9), (10), and (17). □

7. Linear fractional transformation of Weyl–Titchmarsh operators

Here we consider linear fractional transformations of the type

$$M(z) \longrightarrow M_B(z) = (B_{2,1} + B_{2,2}M(z)) \times (B_{1,1} + B_{1,2}M(z))^{-1}, \quad z \in \mathbb{C}_+, \tag{47}$$

where

$$B = (B_{p,q})_{1\leqslant p,q\leqslant 2} \in \mathscr{B}(\mathscr{N} \oplus \mathscr{N}),$$
$$\mathscr{B}(\mathscr{N} \oplus \mathscr{N}) = \{B \in [\mathscr{N} \oplus \mathscr{N}] | B^*\mathscr{J}B = \mathscr{J}\}, \quad \mathscr{J} = \begin{pmatrix} 0 & -I_{\mathscr{N}} \\ I_{\mathscr{N}} & 0 \end{pmatrix}, \tag{48}$$

and $M(z)$ is a Weyl–Titchmarsh operator associated with a π-self-adjoint extension A of $\dot{A}$ in Π_κ. This type of transformations in Hilbert spaces was studied in depth in [8,10].

We present the linear fractional transformation relating the Weyl–Titchmarsh operators $M_{A_\ell,\mathscr{N}_{1,2,+}}$ associated with two π-self-adjoint extensions A_ℓ, $\ell = 1, 2$, of $\dot{A}$.

Theorem 11. *Suppose A_1 and A_2 are π-self-adjoint extensions of $\dot{A}$ and $z \in \rho(A_1) \cap \rho(A_2)$. Then the functions $M_{A_1,\mathscr{N}_+}(z)$ and $M_{A_2,\mathscr{N}_+}(z)$ possess the properties* (23)–(27) *and*

$$M_{A_2,\mathscr{N}_+}(z) = \left(P_{1,2}(\mathrm{i})\big|_{\mathscr{N}_+} + \left(I_{\mathscr{N}_+} + \mathrm{i}P_{1,2}(\mathrm{i})\big|_{\mathscr{N}_+}\right) M_{A_1,\mathscr{N}_+}(z)\right) \times \left(\left(I_{\mathscr{N}_+} + \mathrm{i}P_{1,2}(\mathrm{i})\big|_{\mathscr{N}_+}\right) - P_{1,2}(\mathrm{i})\big|_{\mathscr{N}_+} M_{A_1,\mathscr{N}_+}(z)\right)^{-1}, \tag{49}$$

$$= \mathrm{e}^{-\mathrm{i}\alpha_{1,2}}(\cos(\alpha_{1,2}) + \sin(\alpha_{1,2})M_{A_1,\mathscr{N}_+}(z)) \times (\sin(\alpha_{1,2}) - \cos(\alpha_{1,2})M_{A_1,\mathscr{N}_+}(z))^{-1}\mathrm{e}^{\mathrm{i}\alpha_{1,2}}, \tag{50}$$

where

$$\mathrm{e}^{-2\mathrm{i}\alpha_{1,2}} = -C_{A_2}\, C_{A_1}^{-1}\Big|_{\mathscr{N}_+},$$

$$P_{1,2}(\mathrm{i})\big|_{\mathscr{N}_+} = (\mathrm{i}/2)\left(I - C_{A_2}C_{A_1}^{-1}\right)\Big|_{\mathscr{N}_+}, \tag{51}$$

$$I_{\mathscr{N}_+} + \mathrm{i}P_{1,2}(\mathrm{i})\big|_{\mathscr{N}_+} = (1/2)\left(I + C_{A_2}C_{A_1}^{-1}\right)\Big|_{\mathscr{N}_+}. \tag{52}$$

Proof. Let us assume first that A_1 and A_2 are relatively prime π-self-adjoint extensions of $\dot{A}$. The properties (23)–(27) were proved in Theorem 5. Then using (40) and (42) and following [9] one computes

$$M_{A_2,\mathscr{N}_+}(z) = (zI + (1+z^2)P_{\mathscr{N}_+}(A_2 - z)^{-1}P_{\mathscr{N}_+})\big|_{\mathscr{N}_+}$$
$$= M_{A_1,\mathscr{N}_+}(z) + (1+z^2)P_{\mathscr{N}_+}(A_1 - \mathrm{i})(A_1 - z)^{-1}P_{\mathscr{N}_+} \times (\tan(\alpha_{1,2}) - M_{A_1,\mathscr{N}_+}(z))^{-1}P_{\mathscr{N}_+}(A_1 + \mathrm{i})(A_1 - z)^{-1}P_{\mathscr{N}_+}$$

$$
\begin{aligned}
&= (-\mathrm{i}I_{\mathcal{N}_+} + \tan(\alpha_{1,2}))^{-1}(I_{\mathcal{N}_+} + \tan(\alpha_{1,2})M_{A_1,\mathcal{N}_+}(z)) \\
&\quad \times (\tan(\alpha_{1,2}) - M_{A_1,\mathcal{N}_+}(z))^{-1}((-\mathrm{i}I_{\mathcal{N}_+} + \tan(\alpha_{1,2})) \\
&= \mathrm{e}^{-\mathrm{i}\alpha_{1,2}}(\cos(\alpha_{1,2}) + \sin(\alpha_{1,2})M_{A_1,\mathcal{N}_+}(z))(\sin(\alpha_{1,2}) \\
&\quad - \cos(\alpha_{1,2})M_{A_1,\mathcal{N}_+}(z))^{-1}\mathrm{e}^{\mathrm{i}\alpha_{1,2}}.
\end{aligned}
$$

Eq. (49) then immediately follows from (50) since $P_{1,2}(\mathrm{i})\big|_{\mathcal{N}_+} = (\tan(\alpha_{1,2}) - \mathrm{i}I_{\mathcal{N}_+})^{-1}$ by (19).

Now we treat the general case where the extensions A_1 and A_2 are not relatively prime π-self-adjoint extensions of $\dot{A}$. We choose a π-self-adjoint extension A_3 of $\dot{A}$ such that (A_1, A_3) and (A_2, A_3) are relatively prime w.r.t. $\dot{A}$. (Existence of A_3 can be easily established using the criterion (6)). Then express $M_{A_1,\mathcal{N}_+}(z)$ in terms of $M_{A_3,\mathcal{N}_+}(z)$ and operator $\alpha_{3,1}$ according to (49) and (50) and similarly, express $M_{A_2,\mathcal{N}_+}(z)$ in terms of $M_{A_3,\mathcal{N}_+}(z)$ and some operator $\alpha_{3,2}$. One obtains,

$$
\begin{aligned}
M_{A_1,\mathcal{N}_+}(z) &= \mathrm{e}^{-\mathrm{i}\alpha_{3,1}}(\cos(\alpha_{3,1}) + \sin(\alpha_{3,1})M_{A_3,\mathcal{N}_+}(z)) \\
&\quad \times (\sin(\alpha_{3,1}) - \cos(\alpha_{3,1})M_{A_3,\mathcal{N}_+}(z))^{-1}\mathrm{e}^{\mathrm{i}\alpha_{3,1}},
\end{aligned} \tag{53}
$$

$$
\begin{aligned}
M_{A_2,\mathcal{N}_+}(z) &= \mathrm{e}^{-\mathrm{i}\alpha_{3,2}}(\cos(\alpha_{3,2}) + \sin(\alpha_{3,2})M_{A_3,\mathcal{N}_+}(z)) \\
&\quad \times (\sin(\alpha_{3,2}) - \cos(\alpha_{3,2})M_{A_3,\mathcal{N}_+}(z))^{-1}\mathrm{e}^{\mathrm{i}\alpha_{3,2}}.
\end{aligned} \tag{54}
$$

Computing $M_{A_3,\mathcal{N}_+}(z)$ from (53) yields

$$
\begin{aligned}
M_{A_3,\mathcal{N}_+}(z) &= -\mathrm{e}^{\mathrm{i}\alpha_{3,1}}(\cos(\alpha_{3,1}) - \sin(\alpha_{3,1})M_{A_1,\mathcal{N}_+}(z)) \\
&\quad \times (\sin(\alpha_{3,1}) + \cos(\alpha_{3,1})M_{A_1,\mathcal{N}_+}(z))^{-1}\mathrm{e}^{-\mathrm{i}\alpha_{3,1}}.
\end{aligned} \tag{55}
$$

Insertion of (55) into (54) yields (49)–(50) taking into account (51) and (52). □

A comparison of (50) and (47), (48) then yields

$$
B(\alpha_{1,2}) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\alpha_{1,2}}\sin(\alpha_{1,2}) & -\mathrm{e}^{-\mathrm{i}\alpha_{1,2}}\cos(\alpha_{1,2}) \\ \mathrm{e}^{-\mathrm{i}\alpha_{1,2}}\cos(\alpha_{1,2}) & \mathrm{e}^{-\mathrm{i}\alpha_{1,2}}\sin(\alpha_{1,2}) \end{pmatrix} \in \mathscr{A}(\mathcal{N} \oplus \mathcal{N}) \tag{56}
$$

for the corresponding matrix B in (47) and (48).

8. Example

We conclude with a simple illustration.

Let us define Π_1 as a set of all $L^2([0, 2\pi], \mathrm{d}x)$ functions with the scalar product

$$
[f, g] = \int_0^{2\pi} f(x)\overline{g(x)}\,\mathrm{d}x - \frac{1}{\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x \int_0^{2\pi} \overline{g(x)}\,\mathrm{d}x.
$$

Let also $\dot{A}$ be a π-symmetric operator defined by

$$\dot{A} f = \frac{1}{\mathrm{i}} \frac{\mathrm{d}f}{\mathrm{d}x} \tag{57}$$

with

$$\mathscr{D}(\dot{A}) = \left\{ f \in \Pi_1 \mid f, f' \in AC_{\mathrm{loc}}([0, 2\pi]),\ f(0) = f(2\pi) = 0 \right\}. \tag{58}$$

The corresponding π-adjoint operator $\dot{A}^+$ is then defined by

$$\dot{A}^+ g = \frac{1}{\mathrm{i}} \frac{\mathrm{d}g}{\mathrm{d}x} - \frac{1}{\pi \mathrm{i}} [g(2\pi) - g(0)] \tag{59}$$

with

$$\mathscr{D}(\dot{A}^+) = \left\{ g \in \Pi_1 \mid g, g' \in AC_{\mathrm{loc}}([0, 2\pi]) \right\}. \tag{60}$$

Now one can verify that the deficiency spaces are given by

$$\begin{aligned} \mathscr{N}_\lambda &= \ker(\dot{A}^+ - \lambda) \\ &= \left\{ h(x) \in \mathscr{D}(\dot{A}^+) \mid h(x) = c \cdot \left(\mathrm{e}^{\mathrm{i}\lambda x} - \frac{\mathrm{e}^{2\pi \mathrm{i}\lambda} - 1}{\lambda \pi \mathrm{i}} \right),\ c \in \mathbb{C} \right\}, \end{aligned} \tag{61}$$

and $\dot{A}$ has deficiency indices (1,1). Let us consider a family of π-self-adjoint extensions of $\dot{A}$ parameterized by $\varphi \in (0, 2\pi]$

$$A_\varphi f = \frac{1}{\mathrm{i}} \frac{\mathrm{d}f}{\mathrm{d}x} - \frac{1}{\pi \mathrm{i}} [f(2\pi) - f(0)] \tag{62}$$

with

$$\mathscr{D}(A_\varphi) = \left\{ f \in \mathscr{D}(\dot{A}^+) \mid f(0) + \mathrm{e}^{-\mathrm{i}\varphi} f(2\pi) = 0 \right\}. \tag{63}$$

In order to compute the resolvent $(A_\varphi - \lambda)^{-1}$, we consider

$$(A_\varphi - \lambda) y = f, \quad f \in \mathscr{D}(A_\varphi).$$

This reads

$$\frac{1}{\mathrm{i}} y' - \frac{y(2\pi) - y(0)}{\pi \mathrm{i}} - \lambda y = f(x),$$

and we solve it for $y(x)$

$$y(x) = \mathrm{i}\mathrm{e}^{\mathrm{i}\lambda x} \int_0^x \mathrm{e}^{-\mathrm{i}\lambda t} f(t)\, \mathrm{d}t + \frac{(\mathrm{e}^{\mathrm{i}\varphi} + 1)(1 - \mathrm{e}^{\mathrm{i}\lambda x})}{\lambda \pi \mathrm{i}} C + C \mathrm{e}^{\mathrm{i}\lambda x}.$$

From (62) and (63) follows that the resolvent formula $y(x) = (A_\varphi - \lambda)^{-1} f(x)$ has the form

$$\begin{aligned} y(x) = \mathrm{i}\mathrm{e}^{\mathrm{i}\lambda x} \int_0^x \mathrm{e}^{-\mathrm{i}\lambda t} f(t)\, \mathrm{d}t &+ \frac{\mathrm{i}\mathrm{e}^{2\pi\lambda \mathrm{i}} \left([\mathrm{e}^{\mathrm{i}\varphi} - \pi\lambda \mathrm{i} + 1] \mathrm{e}^{\mathrm{i}\lambda x} - \mathrm{e}^{\mathrm{i}\varphi} - 1 \right)}{(1 + \pi\lambda \mathrm{i}) \mathrm{e}^{\mathrm{i}\varphi} - (1 - \pi\lambda \mathrm{i} + \mathrm{e}^{\mathrm{i}\varphi}) \mathrm{e}^{2\pi\lambda \mathrm{i}} + 1} \\ &\times \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}\lambda t} f(t)\, \mathrm{d}t. \end{aligned}$$

Let us now select two π-self-adjoint extensions $A_1 = A_\pi$ and $A_2 = A_{2\pi}$ of $\dot{A}$. Both extensions are defined by (62) with

$$\mathscr{D}(A_1) = \left\{ f \in \mathscr{D}(\dot{A}^+) \mid f(0) - f(2\pi) = 0 \right\}, \tag{64}$$

$$\mathscr{D}(A_2) = \left\{ f \in \mathscr{D}(\dot{A}^+) \mid f(0) + f(2\pi) = 0 \right\}, \tag{65}$$

respectively. It is clear that A_1 and A_2 are relatively prime with respect to $\dot{A}$. Then

$$(A_1 - \lambda)^{-1} f(x) = \mathrm{i}\mathrm{e}^{\mathrm{i}\lambda x} \int_0^x \mathrm{e}^{-\mathrm{i}\lambda t} f(t)\,\mathrm{d}t - \frac{\mathrm{i}\mathrm{e}^{2\pi\lambda \mathrm{i}}\mathrm{e}^{\mathrm{i}\lambda x}}{\mathrm{e}^{2\pi\lambda\mathrm{i}} - 1} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}\lambda t} f(t)\,\mathrm{d}t \tag{66}$$

and

$$\begin{aligned}(A_2 - \lambda)^{-1} f(x) = \mathrm{i}\mathrm{e}^{\mathrm{i}\lambda x} \int_0^x \mathrm{e}^{-\mathrm{i}\lambda t} f(t)\,\mathrm{d}t + \frac{\mathrm{i}\mathrm{e}^{2\pi\lambda\mathrm{i}}\left([2-\pi\lambda\mathrm{i}]\mathrm{e}^{\mathrm{i}\lambda x} - 2\right)}{\pi\lambda\mathrm{i} - (2-\pi\lambda\mathrm{i})\mathrm{e}^{2\pi\lambda\mathrm{i}} + 2} \\ \times \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}\lambda t} f(t)\,\mathrm{d}t.\end{aligned} \tag{67}$$

A straightforward calculation yields

$$\begin{aligned}&\left((A_2-\lambda)^{-1} - (A_1-\lambda)^{-1}\right) f(x) \\ &\quad = \frac{-2\pi\lambda\mathrm{e}^{2\pi\lambda\mathrm{i}}}{4\mathrm{e}^{2\pi\lambda\mathrm{i}} + \pi\lambda\mathrm{i}(\mathrm{e}^{4\pi\lambda\mathrm{i}} - 1) - 2\mathrm{e}^{4\pi\lambda\mathrm{i}} - 2}\left(\mathrm{e}^{\mathrm{i}\lambda x} - \frac{\mathrm{e}^{2\pi\mathrm{i}\lambda} - 1}{\lambda\pi\mathrm{i}}\right) \\ &\quad\times \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}\lambda t} f(t)\,\mathrm{d}t.\end{aligned} \tag{68}$$

It is easy to see that (68) can be written in the form

$$\left((A_2-\lambda)^{-1} - (A_1-\lambda)^{-1}\right) f(x) = K \cdot [f, g]g, \quad g \in \mathscr{N}_\lambda,$$

where vector $g \in \mathscr{N}_\lambda$ is of the form

$$g = g(x) = \mathrm{e}^{\mathrm{i}\lambda x} - \frac{\mathrm{e}^{2\pi\mathrm{i}\lambda} - 1}{\lambda\pi\mathrm{i}},$$

and the constant K is

$$K = \frac{-2\pi\lambda\mathrm{e}^{2\pi\lambda\mathrm{i}}}{4\mathrm{e}^{2\pi\lambda\mathrm{i}} + \pi\lambda\mathrm{i}(\mathrm{e}^{4\pi\lambda\mathrm{i}} - 1) - 2\mathrm{e}^{4\pi\lambda\mathrm{i}} - 2}.$$

Eq. (68) illustrates the Krein resolvent formula for the π-self-adjoint extensions A_1 and A_2.

Using (67) and (68) we get formulas for $C_{A_1}^{-1}$ and C_{A_2}. Direct computations yield

$$\begin{aligned}C_{A_1}^{-1}\left[\mathrm{e}^{-x} + \frac{\mathrm{e}^{-2\pi} - 1}{\pi}\right] &= -\mathrm{e}^{-2\pi}\left[\mathrm{e}^{x} - \frac{\mathrm{e}^{2\pi} - 1}{\pi}\right], \\ C_{A_2}\left[\mathrm{e}^{x} - \frac{\mathrm{e}^{2\pi} - 1}{\pi}\right] &= \mathrm{e}^{2\pi}\left[\mathrm{e}^{-x} + \frac{\mathrm{e}^{-2\pi} - 1}{\pi}\right].\end{aligned}$$

Therefore,

$$C_{A_2}C_{A_1}^{-1}\Big|_{\mathcal{N}_+} = -\mathrm{e}^{-2\mathrm{i}\alpha_{1,2}}\Big|_{\mathcal{N}_+} = -I_{\mathcal{N}_+},$$

and $\alpha_{1,2} = \pi$. Performing straightforward though tedious calculations we find that

$$P_{12}(z)\Bigg|_{\mathcal{N}_+} = \frac{z(\pi - 2 + (2+\pi)\mathrm{e}^{-2\pi})\left(\mathrm{e}^{2\pi\mathrm{i}z} - 1\right)}{(\pi z\mathrm{i} + 2 + (\pi z\mathrm{i} - 2)\mathrm{e}^{2\pi\mathrm{i}z})(1 - \mathrm{e}^{-2\pi})},$$

and

$$P_{12}(\mathrm{i})\big|_{\mathcal{N}_+} = \mathrm{i}I_{\mathcal{N}_+},$$

that confirms (17). Computing the two functions $M_{A_1,\mathcal{N}_+}(z)$ and $M_{A_2,\mathcal{N}_+}(z)$ we get

$$M_{A_1,\mathcal{N}_+}(z) = \frac{(2 + \pi\mathrm{i}z + (\pi\mathrm{i}z - 2)\mathrm{e}^{2\pi\mathrm{i}z})(1 - \mathrm{e}^{-2\pi})}{z(1 - \mathrm{e}^{2\pi\mathrm{i}z})(\pi - 2 + (\pi + 2)\mathrm{e}^{-2\pi})}, \tag{69}$$

and

$$M_{A_2,\mathcal{N}_+}(z) = \frac{z(\pi - 2 + (2+\pi)\mathrm{e}^{-2\pi})(\mathrm{e}^{2\pi\mathrm{i}z} - 1)}{(\pi z\mathrm{i} + 2 + (\pi z\mathrm{i} - 2)\mathrm{e}^{2\pi\mathrm{i}z})(1 - \mathrm{e}^{-2\pi})}. \tag{70}$$

Direct check confirms that both functions belong to the class N_1 and satisfy properties (23)–(27). Now one can easily verify that for $\alpha_{1,2} = \pi$ we have

$$M_{A_2,\mathcal{N}_+}(z) = -\frac{1}{M_{A_1,\mathcal{N}_+}(z)}.$$

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