

STIELTJES LIKE FUNCTIONS AND INVERSE PROBLEMS FOR SYSTEMS WITH SCHRÖDINGER OPERATOR

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Abstract. A class of scalar Stieltjes like functions is realized as linear-fractional transformations of transfer functions of conservative systems based on a Schrödinger operator T_h in $L_2[a, +\infty)$ with a non-selfadjoint boundary condition. In particular it is shown that any Stieltjes function of this class can be realized in the unique way so that the main operator \mathbb{A} of a system is an accretive $(*)$ -extension of a Schrödinger operator T_h . We derive formulas that restore the system uniquely and allow to find the exact value of a non-real parameter h in the definition of T_h as well as a real parameter μ that appears in the construction of the elements of the realizing system. An elaborate investigation of these formulas shows the dynamics of the restored parameters h and μ in terms of the changing free term γ from the integral representation of the realizable function. It turns out that the parametric equations for the restored parameter h represent different circles whose centers and radii are determined by the realizable function. Similarly, the behavior of the restored parameter μ are described by hyperbolas.

1. Introduction

Realizations of different classes of holomorphic operator-valued functions in the open right half-plane, unit circle, and upper half-plane, as well as inverse spectral problems, play an important role in the spectral analysis of non-self-adjoint operators, interpolation problems, and system theory. The literature on realization theory and inverse spectral problems is too extensive to be discussed exhaustively in this note. We refer, however, to [2], [3], [7], [8], [9], [10], [11], [12], [18], [21], [24], [28] and the literature therein. A class of Herglotz-Nevanlinna functions is a rich source for many types of realization problems. An operator-valued function $V(z)$ acting on a finite-dimensional Hilbert space E belongs to the class of operator-valued Herglotz-Nevanlinna functions if it is holomorphic on $\mathbb{C} \setminus \mathbb{R}$, if it is symmetric with respect to the real axis, i.e., $V(z)^* = V(\bar{z})$, $z \in \mathbb{C} \setminus \mathbb{R}$, and if it satisfies the positivity condition

$$\operatorname{Im} V(z) \geq 0, \quad z \in \mathbb{C}_+.$$

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It is well known (see e.g. [16], [17]) that operator-valued Herglotz-Nevanlinna functions admit the following integral representation:

$$V(z) = Q + Lz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) dG(t), \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{1.1}$$

where $Q = Q^*$, $L \geq 0$, and $G(t)$ is a nondecreasing operator-valued function on \mathbb{R} with values in the class of nonnegative operators in E such that

$$\int_{\mathbb{R}} \frac{(dG(t)x, x)_E}{1+t^2} < \infty, \quad x \in E. \tag{1.2}$$

The realization of a selected class of Herglotz-Nevanlinna functions is provided by a linear conservative system Θ of the form

$$\begin{cases} (\mathbb{A} - zI)x = KJ\varphi_- \\ \varphi_+ = \varphi_- - 2iK^*x \end{cases} \tag{1.3}$$

or

$$\Theta = \left(\begin{array}{cc} \mathbb{A} & K \ J \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & E \end{array} \right). \tag{1.4}$$

In this system \mathbb{A} , the *main operator* of the system, is a so-called $(*)$ -extension, which is a bounded linear operator from \mathcal{H}_+ into \mathcal{H}_- extending a symmetric operator A in \mathcal{H} , where $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ is a rigged Hilbert space. Moreover, K is a bounded linear operator from the finite-dimensional Hilbert space E into \mathcal{H}_- , while $J = J^* = J^{-1}$ is acting on E , are such that $\text{Im } \mathbb{A} = KJK^*$. Also, $\varphi_- \in E$ is an input vector, $\varphi_+ \in E$ is an output vector, and $x \in \mathcal{H}_+$ is a vector of the state space of the system Θ . The system described by (1.3)-(1.4) is called a rigged canonical system of the Livšic type [22] or the Brodskiĭ-Livšic rigged operator colligation, cf., e.g. [11], [12], [13]. The operator-valued function

$$W_{\Theta}(z) = I - 2iK^*(\mathbb{A} - zI)^{-1}KJ \tag{1.5}$$

is a transfer function (or characteristic function) of the system Θ . It was shown in [11] that an operator-valued function $V(z)$ acting on a Hilbert space E of the form (1.1) can be represented and realized in the form

$$V(z) = i[W_{\Theta}(z) + I]^{-1}[W_{\Theta}(z) - I] = K^*(\mathbb{A}_R - zI)^{-1}K, \tag{1.6}$$

where $W_{\Theta}(z)$ is a transfer function of some canonical scattering ($J = I$) system Θ , and where the “*real part*” $\mathbb{A}_R = \frac{1}{2}(\mathbb{A} + \mathbb{A}^*)$ of \mathbb{A} satisfies $\mathbb{A}_R \supset \hat{A} = \hat{A}^* \supset A$ if and only if the function $V(z)$ in (1.1) satisfies the following two conditions:

$$\begin{cases} L = 0, \\ Qx = \int_{\mathbb{R}} \frac{t}{1+t^2} dG(t)x \quad \text{when} \quad \int_{\mathbb{R}} (dG(t)x, x)_E < \infty. \end{cases} \tag{1.7}$$

In the current paper we are going to focus on an important subclass of Herglotz-Nevanlinna functions, the so called Stieltjes like functions that also includes Stieltjes functions. In Section 4 we specify a subclass of realizable Stieltjes operator-functions

and show that any member of this subclass can be realized by a system of the form (1.4) whose main operator \mathbb{A} is accretive.

In Section 5 we introduce a class of Stieltjes like scalar functions. Then we rely on the general realization results developed in Section 4 (see also [15]) to restore a system Θ of the form (1.4) containing the Schrödinger operator in $L_2[a, +\infty)$ with non-self-adjoint boundary conditions

$$\begin{cases} T_h y = -y'' + q(x)y \\ y'(a) = h y(a) \end{cases}, \quad \left(q(x) = \overline{q(x)}, \operatorname{Im} h \neq 0 \right).$$

We show that if a non-decreasing function $\sigma(t)$ is the spectral distribution function of positive self-adjoint boundary value problem

$$\begin{cases} A_\theta y = -y'' + q(x)y \\ y'(a) = \theta y(a) \end{cases}$$

and satisfies conditions

$$\int_0^\infty d\sigma(t) = \infty, \quad \int_0^\infty \frac{d\sigma(t)}{1+t} < \infty,$$

then for every real γ a Stieltjes like function

$$V(z) = \gamma + \int_0^\infty \frac{d\sigma(t)}{t-z}$$

can be realized in the unique way as a $V_\Theta(z)$ function of a rigged canonical system Θ containing some Schrödinger operator T_h . In particular, it is shown that for every $\gamma \geq 0$ a Stieltjes function $V(z)$ with integral representation above can be realized by a system Θ whose main operator \mathbb{A} is an accretive $(*)$ -extension of a Schrödinger operator T_h .

On top of the general realization results, Section 5 provides the reader with formulas that allow to find the exact value of a non-real parameter h in the definition of T_h of the realizing system Θ . Similar investigation is presented in Section 6 to describe the real parameter μ that appears in the construction of the elements of the realizing system. A detailed study of these formulas shows the dynamics of the restored parameters h and μ in terms of a changing free term γ in the integral representation of $V(z)$ above. It will be shown and graphically presented that the parametric equations for the restored parameter h represent different circles whose centers and radii are completely determined by the function $V(z)$. Similarly, the behavior of the restored parameter μ are described by hyperbolas.

2. Some preliminaries

For a pair of Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ we denote by $[\mathcal{H}_1, \mathcal{H}_2]$ the set of all bounded linear operators from \mathcal{H}_1 to \mathcal{H}_2 . Let A be a closed, densely defined, symmetric operator in a Hilbert space \mathcal{H} with inner product $(f, g), f, g \in \mathcal{H}$. Consider the rigged Hilbert space

$$\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-,$$

where $\mathcal{H}_+ = D(A^*)$ and

$$(f, g)_+ = (f, g) + (A^*f, A^*g), \quad f, g \in D(A^*).$$

Note that identifying the space conjugate to \mathcal{H}_\pm with \mathcal{H}_\mp , we get that if $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ then $\mathbb{A}^* \in [\mathcal{H}_+, \mathcal{H}_-]$.

DEFINITION 2.1. An operator $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ is called a self-adjoint bi-extension of a symmetric operator A if $\mathbb{A} = \mathbb{A}^*$, $\mathbb{A} \supset A$, and the operator

$$\widehat{A}f = \mathbb{A}f, \quad f \in D(\widehat{A}) = \{f \in \mathcal{H}_+ : \mathbb{A}f \in \mathcal{H}\}$$

is self-adjoint in \mathcal{H} .

The operator \widehat{A} in the above definition is called a *quasi-kernel* of a self-adjoint bi-extension \mathbb{A} (see [27]).

DEFINITION 2.2. An operator $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ is called a $(*)$ -extension (or correct bi-extension) of an operator T (with non-empty set $\rho(T)$ of regular points) if

$$\mathbb{A} \supset T \supset A, \quad \mathbb{A}^* \supset T^* \supset A$$

and the operator $\mathbb{A}_R = \frac{1}{2}(\mathbb{A} + \mathbb{A}^*)$ is a self-adjoint bi-extension of an operator A .

The existence, description, and analog of von Neumann's formulas for self-adjoint bi-extensions and $(*)$ -extensions were discussed in [27] (see also [4], [5], [11]). For instance, if Φ is an isometric operator from the defect subspace \mathfrak{N}_i of the symmetric operator A onto the defect subspace \mathfrak{N}_{-i} , then the formulas below establish a one-to one correspondence between $(*)$ -extensions of an operator T and Φ

$$\mathbb{A}f = A^*f + iR(\Phi - I)x, \quad \mathbb{A}^*f = A^*f + iR(\Phi - I)y, \quad (2.1)$$

where $x, y \in \mathfrak{N}_i$ are uniquely determined from the conditions

$$f - (\Phi + I)x \in D(T), \quad f - (\Phi + I)y \in D(T^*)$$

and R is the Riesz-Berezanskii operator of the triplet $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ that maps \mathcal{H}_+ isometrically onto \mathcal{H}_- (see [27]). If the symmetric operator A has deficiency indices (n, n) , then formulas (2.1) can be rewritten in the following form

$$\mathbb{A}f = A^*f + \sum_{k=1}^n \Delta_k(f)V_k, \quad \mathbb{A}^*f = A^*f + \sum_{k=1}^n \delta_k(f)V_k, \quad (2.2)$$

where $\{V_j\}_1^n \in \mathcal{H}_-$ is a basis in the subspace $R(\Phi - I)\mathfrak{N}_i$, and $\{\Delta_k\}_1^n, \{\delta_k\}_1^n$, are bounded linear functionals on \mathcal{H}_+ with the properties

$$\Delta_k(f) = 0, \quad \forall f \in D(T), \quad \delta_k(f) = 0, \quad \forall f \in D(T^*). \tag{2.3}$$

Let $\mathcal{H} = L_2[a, +\infty)$ and $l(y) = -y'' + q(x)y$ where q is a real locally summable function. Suppose that the symmetric operator

$$\begin{cases} Ay = -y'' + q(x)y \\ y(a) = y'(a) = 0 \end{cases} \tag{2.4}$$

has deficiency indices $(1, 1)$. Let D^* be the set of functions locally absolutely continuous together with their first derivatives such that $l(y) \in L_2[a, +\infty)$. Consider $\mathcal{H}_+ = D(A^*) = D^*$ with the scalar product

$$(y, z)_+ = \int_a^\infty (y(x)\overline{z(x)} + l(y)\overline{l(z)}) dx, \quad y, z \in D^*.$$

Let

$$\mathcal{H}_+ \subset L_2[a, +\infty) \subset \mathcal{H}_-$$

be the corresponding triplet of Hilbert spaces. Consider operators

$$\begin{cases} T_h y = l(y) = -y'' + q(x)y \\ h y(a) - y'(a) = 0 \end{cases}, \quad \begin{cases} T_h^* y = l(y) = -y'' + q(x)y \\ \bar{h} y(a) - y'(a) = 0 \end{cases}, \tag{2.5}$$

$$\begin{cases} \widehat{A} y = l(y) = -y'' + q(x)y \\ \mu y(a) - y'(a) = 0 \end{cases}, \quad \text{Im } \mu = 0.$$

It is well known [1] that $\widehat{A} = \widehat{A}^*$. The following theorem was proved in [6].

THEOREM 2.3. *The set of all $(*)$ -extensions of a non-self-adjoint Schrödinger operator T_h of the form (2.5) in $L_2[a, +\infty)$ can be represented in the form*

$$\begin{aligned} \mathbb{A}y &= -y'' + q(x)y - \frac{1}{\mu - h} [y'(a) - hy(a)] [\mu\delta(x - a) + \delta'(x - a)], \\ \mathbb{A}^*y &= -y'' + q(x)y - \frac{1}{\mu - \bar{h}} [y'(a) - \bar{h}y(a)] [\mu\delta(x - a) + \delta'(x - a)]. \end{aligned} \tag{2.6}$$

In addition, the formulas (2.6) establish a one-to-one correspondence between the set of all $()$ -extensions of a Schrödinger operator T_h of the form (2.5) and all real numbers $\mu \in [-\infty, +\infty]$.*

DEFINITION 2.4. An operator T with the domain $D(T)$ and $\rho(T) \neq \emptyset$ acting on a Hilbert space \mathcal{H} is called *accretive* if

$$\text{Re}(Tf, f) \geq 0, \quad \forall f \in D(T).$$

DEFINITION 2.5. An accretive operator T is called [20] α -sectorial if there exists a value of $\alpha \in (0, \pi/2)$ such that

$$\cot \alpha |\operatorname{Im}(Tf, f)| \leq \operatorname{Re}(Tf, f), \quad f \in \mathcal{D}(T).$$

An accretive operator is called *extremal accretive* if it is not α -sectorial for any $\alpha \in (0, \pi/2)$.

Consider the symmetric operator A of the form (2.4) with defect indices (1,1), generated by the differential operation $l(y) = -y'' + q(x)y$. Let $\varphi_k(x, \lambda)$ ($k = 1, 2$) be the solutions of the following Cauchy problems:

$$\begin{cases} l(\varphi_1) = \lambda \varphi_1 \\ \varphi_1(a, \lambda) = 0 \\ \varphi_1'(a, \lambda) = 1 \end{cases}, \quad \begin{cases} l(\varphi_2) = \lambda \varphi_2 \\ \varphi_2(a, \lambda) = -1 \\ \varphi_2'(a, \lambda) = 0 \end{cases}.$$

It is well known [1] that there exists a function $m_\infty(\lambda)$ (called the Weyl-Titchmarsh function) for which

$$\varphi(x, \lambda) = \varphi_2(x, \lambda) + m_\infty(\lambda)\varphi_1(x, \lambda)$$

belongs to $L_2[a, +\infty)$.

Suppose that the symmetric operator A of the form (2.4) with deficiency indices (1,1) is nonnegative, i.e., $(Af, f) \geq 0$ for all $f \in D(A)$. It was shown in [25] that the Schrödinger operator T_h of the form (2.5) is accretive if and only if

$$\operatorname{Re} h \geq -m_\infty(-0). \quad (2.7)$$

For real h such that $h \geq -m_\infty(-0)$ we get a description of all nonnegative self-adjoint extensions of an operator A . For $h = -m_\infty(-0)$ the corresponding operator

$$\begin{cases} A_K y = -y'' + q(x)y \\ y'(a) + m_\infty(-0)y(a) = 0 \end{cases} \quad (2.8)$$

is the Kreĭn-von Neumann extension of A and for $h = +\infty$ the corresponding operator

$$\begin{cases} A_F y = -y'' + q(x)y \\ y(a) = 0 \end{cases} \quad (2.9)$$

is the Friedrichs extension of A (see [25], [6]).

3. Rigged canonical systems with Schrödinger operator

Let \mathbb{A} be $(*)$ -extension of an operator T , i.e.,

$$\mathbb{A} \supset T \supset A, \quad \mathbb{A}^* \supset T^* \supset A$$

where A is a symmetric operator with deficiency indices (n, n) and $D(A) = D(T) \cap D(T^*)$. In what follows we will only consider the case when the symmetric operator A has dense domain, i.e., $\overline{\mathcal{D}(A)} = \mathcal{H}$.

DEFINITION 3.1. A system of equations

$$\begin{cases} (\mathbb{A} - zI)x = KJ\varphi_- \\ \varphi_+ = \varphi_- - 2iK^*x \end{cases},$$

or an array

$$\Theta = \begin{pmatrix} \mathbb{A} & K & J \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & E \end{pmatrix} \quad (3.1)$$

is called a *rigged canonical system of the Livsic type* or the *Brodskii-Livsic rigged operator colligation* if:

- 1) E is a finite-dimensional Hilbert space with scalar product $(\cdot, \cdot)_E$ and the operator J in this space satisfies the conditions $J = J^* = J^{-1}$,
- 2) $K \in [E, \mathcal{H}_-]$, $\ker K = \{0\}$,
- 3) $\text{Im } \mathbb{A} = KJK^*$, where $K^* \in [\mathcal{H}_+, E]$ is the adjoint of K .

In the definition above $\varphi_- \in E$ stands for an input vector, $\varphi_+ \in E$ is an output vector, and x is a state space vector in \mathcal{H} . An operator \mathbb{A} is called a *main operator* of the system Θ , J is a *direction operator*, and K is a *channel operator*. An operator-valued function

$$W_\Theta(\lambda) = I - 2iK^*(\mathbb{A} - \lambda I)^{-1}KJ \quad (3.2)$$

defined on the set $\rho(T)$ of regular points of an operator T is called the *transfer function* (*characteristic function*) of the system Θ , i.e., $\varphi_+ = W_\Theta(\lambda)\varphi_-$. It is known [25],[27] that any $(*)$ -extension \mathbb{A} of an operator T ($A^* \supset T \supset A$), where A is a symmetric operator with deficiency indices (n, n) ($n < \infty$), $D(A) = D(T) \cap D(T^*)$, can be included as a main operator of some rigged canonical system with $\dim E < \infty$ and invertible channel operator K .

It was also established [25], [27] that

$$V_\Theta(\lambda) = K^*(\text{Re } \mathbb{A} - \lambda I)^{-1}K \quad (3.3)$$

is a Herglotz-Nevanlinna operator-valued function acting on a Hilbert space E , satisfying the following relation for $\lambda \in \rho(T)$, $\text{Im } \lambda \neq 0$

$$V_\Theta(\lambda) = i[W_\Theta(\lambda) - I][W_\Theta(\lambda) + I]^{-1}J. \quad (3.4)$$

Alternatively,

$$\begin{aligned} W_\Theta(\lambda) &= (I + iV_\Theta(\lambda)J)^{-1}(I - iV_\Theta(\lambda)J) \\ &= (I - iV_\Theta(\lambda)J)(I + iV_\Theta(\lambda)J)^{-1}. \end{aligned} \quad (3.5)$$

Let us recall (see [27], [6]) that a symmetric operator with dense domain $\mathcal{D}(A)$ is called *prime* if there is no reducing, nontrivial invariant subspace on which A induces a self-adjoint operator. It was established in [26] that a symmetric operator A is prime if and only if

$$c.l.s. \mathfrak{N}_\lambda = \mathcal{H} \quad (\lambda \neq \bar{\lambda}) \quad (3.6)$$

We call a rigged canonical system of the form (3.1) *prime* if

$$c.l.s. \mathfrak{N}_\lambda = \mathcal{H} \quad (\lambda \neq \bar{\lambda}, \lambda \in \rho(T))$$

One easily verifies that if system Θ is prime, then a symmetric operator A of the system is prime as well.

The following theorem [6] establishes the connection between two rigged canonical systems with equal transfer functions.

THEOREM 3.2. Let $\Theta_1 = \begin{pmatrix} \mathbb{A}_1 & K_1 & J \\ \mathcal{H}_{+1} \subset \mathcal{H}_1 \subset \mathcal{H}_{-1} & & E \end{pmatrix}$ and

$\Theta_2 = \begin{pmatrix} \mathbb{A}_2 & K_2 & J \\ \mathcal{H}_{+2} \subset \mathcal{H}_2 \subset \mathcal{H}_{-2} & & E \end{pmatrix}$ be two prime rigged canonical systems of the Livsic type with

$$\begin{aligned} \mathbb{A}_1 \supset T_1 \supset A_1, \quad \mathbb{A}_1^* \supset T_1^* \supset A_1, \\ \mathbb{A}_2 \supset T_2 \supset A_2, \quad \mathbb{A}_2^* \supset T_2^* \supset A_2, \end{aligned} \quad (3.7)$$

and such that A_1 and A_2 have finite and equal defect indices.

If

$$W_{\Theta_1}(\lambda) = W_{\Theta_2}(\lambda), \quad (3.8)$$

then there exists an isometric operator U from \mathcal{H}_1 onto \mathcal{H}_2 such that $U_+ = U|_{\mathcal{H}_{+1}}$ is an isometry¹ from \mathcal{H}_{+1} onto \mathcal{H}_{+2} , $U_-^* = U_+^*$ is an isometry from \mathcal{H}_{-1} onto \mathcal{H}_{-2} , and

$$UT_1 = T_2U, \quad \mathbb{A}_2 = U_- \mathbb{A}_1 U_+^{-1}, \quad U_- K_1 = K_2. \quad (3.9)$$

COROLLARY 3.3. Let Θ_1 and Θ_2 be the two prime systems from the statement of theorem 3.2. Then the mapping U described in the conclusion of the theorem is unique.

Proof. First let us make an observation that if $\Theta = \begin{pmatrix} \mathbb{A} & K & J \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & E \end{pmatrix}$

is a prime rigged canonical system such that $U_- \mathbb{A} = \mathbb{A} U_+$ and $U_- K = K$, where U is an isometry mapping described in theorem 3.2, then $U = I$. Indeed, it is well known [27] that

$$(\operatorname{Re} \mathbb{A} - \lambda I)^{-1} K E = \mathfrak{N}_\lambda. \quad (3.10)$$

We have

$$\begin{aligned} U(\operatorname{Re} \mathbb{A} - \lambda I)^{-1} K e &= U_+(\operatorname{Re} \mathbb{A} - \lambda I)^{-1} K e \\ &= (\operatorname{Re} \mathbb{A} - \lambda I)^{-1} U_- K e \\ &= (\operatorname{Re} \mathbb{A} - \lambda I)^{-1} K e, \quad \forall e \in E, \lambda \neq \bar{\lambda}. \end{aligned}$$

Combining the above equation with (3.6) and (3.10) we obtain $U = I$.

Now let Θ_1 and Θ_2 be the two prime systems from the statement of theorem 3.2. Suppose there are two isometric mappings U_1 and U_2 guaranteed by theorem 3.2. Then the relations

$$\mathbb{A}_2 = U_{-,1} \mathbb{A}_1 U_{+,1}^{-1}, \quad U_{-,1} K_1 = K_2, \quad \mathbb{A}_2 = U_{-,2} \mathbb{A}_1 U_{+,2}^{-1}, \quad U_{-,2} K_1 = K_2,$$

lead to

$$\mathbb{A}_1 U_{+,1}^{-1} U_{+,2} = U_{-,1}^{-1} U_{-,2} \mathbb{A}_1, \quad U_{-,1}^{-1} U_{-,2} K = K.$$

¹It was shown in [6] that the operator U_+ defined this way is an isometry from \mathcal{H}_{+1} onto \mathcal{H}_{+2} . It is also shown there that the isometric operator $U^* : \mathcal{H}_{+2} \rightarrow \mathcal{H}_{+1}$ uniquely defines operator $U_- = (U^*)^* : \mathcal{H}_{-1} \rightarrow \mathcal{H}_{-2}$.

Since Θ_1 is prime then $U_1^{-1}U_2 = I$ and hence $U_1 = U_2$. This proves the uniqueness of U . \square

Now we shall construct a rigged canonical system based on a non-self-adjoint Schrödinger operator. One can easily check that the $(*)$ -extension

$$\mathbb{A}y = -y'' + q(x)y - \frac{1}{\mu - h} [y'(a) - hy(a)] [\mu\delta(x - a) + \delta'(x - a)], \quad \text{Im } h > 0$$

of the non-self-adjoint Schrödinger operator T_h of the form (2.5) satisfies the condition

$$\text{Im } \mathbb{A} = \frac{\mathbb{A} - \mathbb{A}^*}{2i} = (\cdot, g)g, \tag{3.11}$$

where

$$g = \frac{(\text{Im } h)^{\frac{1}{2}}}{|\mu - h|} [\mu\delta(x - a) + \delta'(x - a)] \tag{3.12}$$

and $\delta(x - a)$, $\delta'(x - a)$ are the delta-function and its derivative at the point a . Moreover,

$$(y, g) = \frac{(\text{Im } h)^{\frac{1}{2}}}{|\mu - h|} [\mu y(a) - y'(a)], \tag{3.13}$$

where

$$y \in \mathcal{H}_+, g \in \mathcal{H}_-, \mathcal{H}_+ \subset L_2(a, +\infty) \subset \mathcal{H}_-$$

and the triplet of Hilbert spaces is as discussed in theorem 2.3. Let $E = \mathbb{C}$, $Kc = cg$ ($c \in \mathbb{C}$). It is clear that

$$K^*y = (y, g), \quad y \in \mathcal{H}_+ \tag{3.14}$$

and $\text{Im } \mathbb{A} = KK^*$. Therefore, the array

$$\Theta = \begin{pmatrix} \mathbb{A} & K & 1 \\ \mathcal{H}_+ \subset L_2[a, +\infty) \subset \mathcal{H}_- & \mathbb{C} & \end{pmatrix} \tag{3.15}$$

is a rigged canonical system with the main operator \mathbb{A} of the form (2.6), the direction operator $J = 1$ and the channel operator K of the form (3.14). Our next logical step is finding the transfer function of (3.15). It was shown in [6] that

$$W_{\Theta}(\lambda) = \frac{\mu - h}{\mu - \bar{h}} \frac{m_{\infty}(\lambda) + \bar{h}}{m_{\infty}(\lambda) + h}, \tag{3.16}$$

and

$$V_{\Theta}(\lambda) = \frac{(m_{\infty}(\lambda) + \mu) \text{Im } h}{(\mu - \text{Re } h) m_{\infty}(\lambda) + \mu \text{Re } h - |h|^2}. \tag{3.17}$$

4. Realization of Stieltjes functions

Let E be a finite-dimensional Hilbert space. The scalar versions of the following definition can be found in [19].

DEFINITION 4.1. We will call an operator-valued Herglotz-Nevalinna function $V(z) \in [E, E]$ by a *Stieltjes function* if $V(z)$ admits the following integral representation

$$V(z) = \gamma + \int_0^{\infty} \frac{dG(t)}{t - z}, \quad (4.1)$$

where $\gamma \geq 0$ and $G(t)$ is a non-decreasing on $[0, +\infty)$ operator-valued function such that

$$\int_0^{\infty} \frac{(dG(t)e, e)_E}{1 + t} < \infty, \quad \forall e \in E.$$

Alternatively (see [19]) an operator-valued function $V(z)$ is Stieltjes if it is holomorphic in $\text{Ext}[0, +\infty)$ and

$$\frac{\text{Im}[zV(z)]}{\text{Im} z} \geq 0. \quad (4.2)$$

The theorem 4.2 below was stated in [14], [15] and we present its proof for the convenience of a reader.

THEOREM 4.2. *Let Θ be a prime system of the form (3.1). Then an operator-valued function $V_{\Theta}(z)$ defined by (3.3), (3.4) is a Stieltjes function if and only if the main operator \mathbb{A} of the system Θ is accretive.*

Proof. Let us assume first that \mathbb{A} is an accretive operator, i.e. $(\text{Re } \mathbb{A}x, x) \geq 0$, for all $x \in \mathcal{H}_+$. Let $\{z_k\}$ ($k = 1, \dots, n$) be a sequence of non-real complex numbers and h_k be a sequence of vectors in E . Let us denote

$$Kh_k = \delta_k, \quad x_k = (\text{Re } \mathbb{A} - z_k I)^{-1} \delta_k, \quad x = \sum_{k=1}^n x_k. \quad (4.3)$$

Since $(\text{Re } \mathbb{A}x, x) \geq 0$, we have

$$\sum_{k,l=1}^n (\text{Re } \mathbb{A}x_k, x_l) \geq 0. \quad (4.4)$$

By formal calculations one can have

$$\text{Re } \mathbb{A}x_k = \delta_k + z_k (\text{Re } \mathbb{A} - z_k I)^{-1} \delta_k,$$

and

$$\sum_{k,l=1}^n (\text{Re } \mathbb{A}x_k, x_l) = \sum_{k,l=1}^n [(\delta_k, (\text{Re } \mathbb{A} - z_l I)^{-1} \delta_l) + (z_k (\text{Re } \mathbb{A} - z_k I)^{-1} \delta_k, (\text{Re } \mathbb{A} - z_l I)^{-1} \delta_l)].$$

Using obvious equalities

$$((\operatorname{Re} \mathbb{A} - z_k I)^{-1} K h_k, K h_l) = (V_\theta(z_k) h_k, h_l)_E,$$

and

$$((\operatorname{Re} \mathbb{A} - \bar{z}_l I)^{-1} (\operatorname{Re} \mathbb{A} - z_k I)^{-1} K h_k, K h_l) = \left(\frac{V_\theta(z_k) - V_\theta(\bar{z}_l)}{z_k - \bar{z}_l} h_k, h_l \right)_E,$$

we obtain

$$\sum_{k,l=1}^n (\operatorname{Re} \mathbb{A} x_k, x_l) = \sum_{k,l=1}^n \left(\frac{z_k V_\theta(z_k) - \bar{z}_l V_\theta(\bar{z}_l)}{z_k - \bar{z}_l} h_k, h_l \right)_E \geq 0. \tag{4.5}$$

The choice of z_k was arbitrary, which means that $V_\theta(z)$ is a Stieltjes function (see [3]).

Now we prove necessity. Since Θ is a prime system then A is a prime symmetric operator. Then the equivalence of (4.5) and (4.4) implies that $(\operatorname{Re} \mathbb{A} x, x) \geq 0$ for any x from c.l.s. $\{\mathfrak{N}_z\}$, $z \neq \bar{z}$. As we have already mentioned above, a symmetric operator A with the equal deficiency indices is prime if and only if for all $\lambda \neq \bar{\lambda}$

$$\text{c.l.s. } \{\mathfrak{N}_\lambda\} = \mathcal{H}.$$

Therefore we can conclude that $(\operatorname{Re} \mathbb{A} x, x) \geq 0$ for any $x \in \mathcal{H}_+$ and hence \mathbb{A} is an accretive operator. □

A system Θ of the form (3.1) is called an *accretive system* if its main operator \mathbb{A} is accretive.

Now we define a certain class $S_0(R)$ of realizable Stieltjes functions. At this point we need to note that since Stieltjes functions form a subset of Herglotz-Nevanlinna functions then we can utilize the conditions (1.7) to form a *class* $S(R)$ of all *realizable Stieltjes functions* (see also [15]). Clearly, $S(R)$ is a subclass of $N(R)$ of all realizable Herglotz-Nevanlinna functions described in details in [11] and [12]. To see the specifications of the class $S(R)$ we recall that aside of integral representation (4.1), any Stieltjes function admits a representation (1.1). Applying condition (1.7) we obtain

$$Q = \frac{1}{2} [V_\theta(-i) + V_\theta^*(-i)] = \gamma + \int_0^{+\infty} \frac{t}{1+t^2} dG(t). \tag{4.6}$$

Combining the second part of condition (1.7) and (4.6) we conclude that

$$\gamma e = 0, \tag{4.7}$$

for all $e \in E$ such that

$$\int_0^\infty (dG(t)e, e)_E < \infty. \tag{4.8}$$

holds. Consequently, (4.7)-(4.8) is precisely the condition for $V(z) \in S(R)$.

We are going to focus though on the subclass $S_0(R)$ of $S(R)$ whose definition is the following.

DEFINITION 4.3. An operator-valued Stieltjes function $V(z) \in [E, E]$ is said to be a member of the class $S_0(\mathcal{R})$ if in the representation (4.1) we have

$$\int_0^\infty (dG(t)e, e)_E = \infty. \tag{4.9}$$

for all non-zero $e \in E$.

We note that a function $V(z)$ can belong to class $S_0(\mathcal{R})$ and have an arbitrary constant $\gamma \geq 0$ in the representation (4.1).

The following statement [15] is the direct realization theorem for the functions of the class $S_0(\mathcal{R})$.

THEOREM 4.4. *Let Θ be an accretive system of the form (3.1). Then the operator-function $V_\Theta(z)$ of the form (3.3), (3.4) belongs to the class $S_0(\mathcal{R})$.*

Proof. To see that $V_\Theta(z)$ is a Stieltjes operator-function we merely apply theorem 4.2 to system Θ .

Now we will show that $V_\Theta(z)$ belongs to $S_0(\mathcal{R})$. It was shown in [11] and [12] that $E_\infty = K^{-1}\mathcal{L}$, where $\mathcal{L} = \mathcal{H} \ominus \overline{\mathcal{D}(A)}$ and

$$E_\infty = \left\{ e \in E : \int_0^{+\infty} (dG(t)e, e)_E < \infty \right\}.$$

But $\overline{\mathcal{D}(A)} = \mathcal{H}$ and consequently $\mathcal{L} = \{0\}$. Next, $E_\infty = \{0\}$,

$$\int_0^\infty (dG(t)e, e)_E = \infty,$$

for all non-zero $e \in E$, and therefore $V_\Theta(z) \in S_0(\mathcal{R})$. □

The inverse realization theorem can be stated and proved (see [15]) for the classes $S_0(\mathcal{R})$ as follows.

THEOREM 4.5. *Let an operator-valued function $V(z)$ belong to the class $S_0(\mathcal{R})$. Then $V(z)$ admits a realization by an accretive prime system Θ of the form (3.1) with $\mathcal{D}(T) \neq \mathcal{D}(T^*)$ and $J = I$.*

Proof. We have already noted that the class of Stieltjes function lies inside the wider class of all Herglotz-Nevanlinna functions. Thus all we actually have to show is that $S_0(\mathcal{R}) \subset N_0(\mathcal{R})$, where $N_0(\mathcal{R})$ is subclass of realizable Herglotz-Nevanlinna functions described in [12], and that the realizing system constructed in [12] appears to be an accretive system. The former is rather obvious and follows directly from the definition of the class $S_0(\mathcal{R})$. To see that the realizing system is accretive we need to apply theorem 4.2 to $V_\Theta(z) = V(z)$, where $V_\Theta(z)$ is related to the model system Θ that was constructed in [12]. As it was also shown in [11] and [12], the symmetric operator A of the model system Θ is prime and hence (3.6) takes place. We are going to show that in this case the system Θ is also prime, i.e.,

$$\underset{\lambda \neq \bar{\lambda}, \lambda \in \rho(T)}{c.l.s.} \mathfrak{N}_\lambda = \mathcal{H}. \tag{4.10}$$

Consider the operator $U_{\lambda_0\lambda} = (\tilde{A} - \lambda_0 I)(\tilde{A} - \lambda I)^{-1}$, where \tilde{A} is an arbitrary self-adjoint extension of A . By a simple check one confirms that $U_{\lambda_0\lambda} \mathfrak{N}_{\lambda_0} = \mathfrak{N}_\lambda$. To prove (4.10) we assume that there is a function $f \in \mathcal{H}$ such that

$$f \perp_{c.l.s.} \mathfrak{N}_\lambda, \quad \lambda \neq \bar{\lambda}, \lambda \in \rho(T)$$

Then $(f, U_{\lambda_0\lambda} g) = 0$ for all $g \in \mathfrak{N}_{\lambda_0}$ and all $\lambda \in \rho(T)$. But accretiveness of the system Θ implies that there are regular points of T in the upper and lower half-planes. This leads to a conclusion that the function $\phi(\lambda) = (f, U_{\lambda_0\lambda} g) \equiv 0$ for all $\lambda \neq \bar{\lambda}$. Combining this with (3.6) we conclude that $f = 0$ and thus (4.10) holds. \square

5. Restoring a non-self-adjoint Schrödinger operator T_h

In this section we are going to use the realization results for Stieltjes functions developed in section 4 to obtain the solution of inverse spectral problem for Schrödinger operator of the form (2.5) in $L_2[a, +\infty)$ with non-self-adjoint boundary conditions

$$\begin{cases} T_h y = -y'' + q(x)y \\ y'(a) = h y(a) \end{cases}, \quad \left(q(x) = \overline{q(x)}, \operatorname{Im} h \neq 0 \right). \quad (5.1)$$

In particular, we will show that if a non-decreasing function $\sigma(t)$ is the spectral function of positive self-adjoint boundary value problem

$$\begin{cases} A_\theta y = -y'' + q(x)y \\ y'(a) = \theta y(a) \end{cases} \quad (5.2)$$

and satisfies conditions

$$\int_0^\infty d\sigma(t) = \infty, \quad \int_0^\infty \frac{d\sigma(t)}{1+t} < \infty, \quad (5.3)$$

then for every $\gamma \geq 0$ a Stieltjes function

$$V(z) = \gamma + \int_0^\infty \frac{d\sigma(t)}{t-z}$$

can be realized in the unique way as a $V_\Theta(z)$ function of an accretive rigged canonical system Θ with some Schrödinger operator T_h .

Let $\mathcal{H} = L_2[a, +\infty)$ and $l(y) = -y'' + q(x)y$ where q is a real locally summable function. We consider a symmetric operator with defect indices $(1, 1)$

$$\begin{cases} \tilde{B}y = -y'' + q(x)y \\ y'(a) = y(a) = 0 \end{cases} \quad (5.4)$$

together with its positive self-adjoint extension of the form

$$\begin{cases} \tilde{B}_\theta y = -y'' + q(x)y \\ y'(a) = \theta y(a) \end{cases} \quad (5.5)$$

defined in $\mathcal{H} = L_2[a, +\infty)$. A non-decreasing function $\sigma(\lambda)$ defined on $[0, +\infty)$ is called the *distribution function* (see [23]) of an operator pair $\tilde{B}_\theta, \tilde{B}$, where \tilde{B}_θ of the form (5.5) is a self-adjoint extension of symmetric operator \tilde{B} of the form (5.4), and if the formulas

$$\begin{aligned} \varphi(\lambda) &= Uf(x), \\ f(x) &= U^{-1}\varphi(\lambda), \end{aligned} \tag{5.6}$$

establish one-to-one isometric correspondence U between $L_2^\sigma[0, +\infty)$ and $L_2[a, +\infty)$. Moreover, this correspondence is such that the operator \tilde{B}_θ is unitarily equivalent to the operator

$$\Lambda_\sigma\varphi(\lambda) = \lambda\varphi(\lambda), \quad (\varphi(\lambda) \in L_2^\sigma[0, +\infty)) \tag{5.7}$$

in $L_2^\sigma[0, +\infty)$ while symmetric operator \tilde{B} in (5.4) is unitarily equivalent to the symmetric operator

$$\Lambda_\sigma^0\varphi(\lambda) = \lambda\varphi(\lambda), \quad D(\Lambda_\sigma^0) = \left\{ \varphi(\lambda) \in L_2^\sigma[0, +\infty) : \int_0^{+\infty} \varphi(\lambda)d\sigma(\lambda) = 0 \right\}. \tag{5.8}$$

DEFINITION 5.1. A scalar Herglotz-Nevanlinna function $V(z)$ is called *Stieltjes like function* if it has an integral representation (4.1) with an arbitrary (not necessarily non-negative) constant γ .

We are going to introduce a new class of realizable scalar Stieltjes like functions whose structure is similar to that of $S_0(R)$ of section 4.

DEFINITION 5.2. A Stieltjes like function $V(z)$ is said to be a member of the class $SL_0(R)$ if it admits an integral representation

$$V(z) = \gamma + \int_0^\infty \frac{d\sigma(t)}{t-z}, \quad (\gamma \in (-\infty, +\infty)), \tag{5.9}$$

where non-decreasing function $\sigma(t)$ satisfies the following conditions

$$\int_0^\infty d\sigma(t) = \infty, \quad \int_0^\infty \frac{d\sigma(t)}{1+t} < \infty. \tag{5.10}$$

Consider the following subclasses of $SL_0(R)$.

DEFINITION 5.3. A function $V(z) \in SL_0(R)$ belongs to the class $SL_0^K(R)$ if

$$\int_0^\infty \frac{d\sigma(t)}{t} = \infty. \tag{5.11}$$

DEFINITION 5.4. A function $V(z) \in SL_0(R)$ belongs to the class $SL_{01}^K(R)$ if

$$\int_0^\infty \frac{d\sigma(t)}{t} < \infty. \tag{5.12}$$

The following theorem describes the realization of the class $SL_0(R)$.

THEOREM 5.5. *Let $V(z) \in SL_0(R)$ and the function $\sigma(t)$ be the distribution function of an operator pair \tilde{B}_θ of the form (5.4) and \tilde{B} of the form (5.5). Then there exist unique Schrödinger operator T_h ($\text{Im} h > 0$) of the form (5.1), operator \mathbb{A} given by (2.6), operator K as in (3.14), and the rigged canonical system of the Livsic type*

$$\Theta = \left(\begin{array}{ccc} \mathbb{A} & K & 1 \\ \mathcal{H}_+ \subset L_2[a, +\infty) \subset \mathcal{H}_- & & \mathbb{C} \end{array} \right), \quad (5.13)$$

of the form (3.15) so that $V(z)$ is realized by Θ .

Proof. Since $\sigma(t)$ is the distribution function of the positive self-adjoint operator, then (see [23]) we can completely restore the operator \tilde{B}_θ of the form (5.5) as well as a symmetric operator \tilde{B} of the form (5.4). It follows from the definition of the distribution function above that there is operator U defined in (5.6) establishing one-to-one isometric correspondence between $L_2^\sigma[0, +\infty)$ and $L_2[a, +\infty)$ while providing for the unitary equivalence between the operator \tilde{B}_θ and operator of multiplication by independent variable Λ_σ of the form (5.7). Taking this into account, we realize (see [11]) a Herglotz-Nevanlinna function $V(z)$ with a rigged canonical system

$$\Theta_\Lambda = \left(\begin{array}{ccc} \Lambda & K^\sigma & 1 \\ \mathcal{H}_+^\sigma \subset L_2^\sigma[0, +\infty) \subset \mathcal{H}_-^\sigma & & \mathbb{C} \end{array} \right).$$

Following the steps for construction of the model system described in [11], we note that

$$\Lambda = \text{Re } \Lambda + iK^\sigma(K^\sigma)^*$$

is a correct $(*)$ -extension of an operator T^σ such that $\Lambda \supset T^\sigma \supset \Lambda_\sigma^0$ where Λ_σ^0 is defined in (5.8). The real part $\text{Re } \Lambda$ is a self-adjoint bi-extension of Λ_σ^0 that has a quasi-kernel Λ_σ of the form (5.7). The operator K^σ in the above system is defined by

$$K^\sigma c = c \cdot \alpha, \quad c \in \mathbb{C}, \quad \alpha \in \mathcal{H}_-^\sigma.$$

In addition we can observe that the function $\eta(\lambda) \equiv 1$ belongs to the space \mathcal{H}_-^σ . To confirm this we need to show that $(x, 1)$ defines a continuous linear functional for every $x \in \mathcal{H}_+^\sigma$. It was shown in [11], [12] that

$$\mathcal{H}_+^\sigma = \mathcal{D}(\Lambda_\sigma^0) \dot{+} \left\{ \frac{c_1}{1+t^2} \right\} \dot{+} \left\{ \frac{c_2 t}{1+t^2} \right\}, \quad c_1, c_2 \in \mathbb{C}.$$

Consequently, every vector $x \in \mathcal{H}_+^\sigma$ has three components $x = x_1 + x_2 + x_3$ according to the decomposition above. Obviously, $(x_1, 1)$ and $(x_2, 1)$ yield convergent integrals while $(x_3, 1)$ boils down to

$$\int_0^\infty \frac{t}{1+t^2} d\sigma(t).$$

To see the convergence of the above integral we notice that

$$\frac{t}{1+t^2} = \frac{t-1}{(1+t^2)(t+1)} + \frac{1}{1+t} \leq \frac{1}{1+t^2} + \frac{1}{1+t}.$$

The integrals taken of the last two expressions on the right side converge due to (1.2) and (5.10), and hence so does the integral of the left side. Thus, $(x, 1)$ defines a continuous linear functional for every $x \in \mathcal{H}_+^\sigma$, and hence $1 \in \mathcal{H}^\sigma$.

The state space of the system Θ_λ is $\mathcal{H}_+^\sigma \subset L_2^\sigma[0, +\infty) \subset \mathcal{H}_-^\sigma$, where $\mathcal{H}_+^\sigma = \mathcal{D}((\Lambda_\sigma^0)^*)$. By the realization theorem [11] we have that $V(z) = V_{\Theta_\lambda}(z)$.

We can also show that the system Θ_λ is a prime system. In order to do so we need to show that

$$c.l.s. \quad \mathfrak{N}_\lambda = L_2^\sigma[0, +\infty), \tag{5.14}$$

$$\lambda \neq \bar{\lambda}, \lambda \in \rho(T^\sigma)$$

where \mathfrak{N}_λ are defect subspaces of the symmetric operator Λ_σ^0 . It is known (see [11]) that Λ_σ^0 is a prime operator. Hence we can follow the reasoning of the proof of theorem 4.5 and only confirm that operator T^σ has regular points in the upper and lower half-planes. To see this we first note that non-negative operator Λ_σ^0 has no kernel spectrum [1] on the left real half-axis. Then we apply Theorem 1 of [1] (see page 149 of vol. 2 of [1]) that gives the complete description of the spectrum of T^σ . This theorem implies that there are regular points of T^σ on the left real half-axis. Since $\rho(T^\sigma)$ is an open set we confirm the presence of non-real regular points of T^σ in both half-planes. Thus (5.14) holds and Θ_λ is a prime system.

Applying theorem 3.2 on unitary equivalence to the isometry U defined in (5.6) we obtain a triplet of isometric operators U_+, U , and U_- , where

$$U_+ = U|_{\mathcal{H}_+^\sigma}, \quad U_-^* = U_+^*.$$

This triplet of isometric operators will map the rigged space $\mathcal{H}_+^\sigma \subset L_2^\sigma[0, +\infty) \subset \mathcal{H}_-^\sigma$ into another rigged Hilbert space $\mathcal{H}_+ \subset L_2^\sigma[a, +\infty) \subset \mathcal{H}_-$. Moreover, U_+ is an isometry from $\mathcal{H}_+^\sigma = \mathcal{D}(\Lambda_\sigma^{0*})$ onto $\mathcal{H}_+ = \mathcal{D}(\tilde{B}^*)$, and $U_-^* = U_+^*$ is an isometry from \mathcal{H}_+^σ onto \mathcal{H}_- . This is true since the operator U provides the unitary equivalence between the symmetric operators \tilde{B} and Λ_σ^0 .

Now we construct a system

$$\Theta = \begin{pmatrix} \mathbb{A} & K & 1 \\ \mathcal{H}_+ \subset L_2[a, +\infty) \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix}$$

where $K = U_-K^\sigma$ and $\mathbb{A} = U_- \Lambda U_+^{-1}$ is a correct $(*)$ -extension of operator $T = UT^\sigma U^{-1}$ such that $\mathbb{A} \supset T \supset \tilde{B}$. The real part $\text{Re } \mathbb{A}$ contains the quasi-kernel \tilde{B}_θ . This construction of \mathbb{A} is unique due to the theorem on the uniqueness of a $(*)$ -extension for a given quasi-kernel (see [27]). On the other hand, all $(*)$ -extensions based on a pair $\tilde{B}, \tilde{B}_\theta$ must take form (2.6) for some values of parameters h and μ . Consequently, our function $V(z)$ is realized by the system Θ of the form (5.13) and

$$V(z) = V_{\Theta_\lambda}(z) = V_\Theta(z).$$

□

REMARK 5.6. Applying corollary 3.3 to the mapping U defined by (5.6) we obtain that the operator U in the above theorem is unique. The uniqueness of the operator U leads to an interesting observation. Let $u_k(x, \lambda)$, $(k = 1, 2)$ be the solutions of the following Cauchy problems:

$$\begin{cases} l(u_1) = \lambda u_1 \\ u_1(a, \lambda) = 0 \\ u_1'(a, \lambda) = 1 \end{cases}, \quad \begin{cases} l(u_2) = \lambda u_2 \\ u_2(a, \lambda) = 1 \\ u_2'(a, \lambda) = 0 \end{cases}.$$

Traditionally, (see [23]) a non-decreasing function $\sigma(\lambda)$ defined on $[0, +\infty)$ is called the *distribution function of a self-adjoint operator* \tilde{B}_θ of the form (5.5) if the formulas

$$\begin{aligned} \varphi(\lambda) &= Uf(x) = \int_a^{+\infty} f(x)u(x, \lambda) dx, \\ f(x) &= U^{-1}\varphi(\lambda) = \int_0^{+\infty} \varphi(\lambda)u(x, \lambda) d\sigma(\lambda), \end{aligned} \quad (5.15)$$

where $u(x, \lambda) = u_1(x, \lambda) + \theta u_2(x, \lambda)$, establish one-to-one isometric correspondence U between $L_2^\sigma[0, +\infty)$ and $L_2[a, +\infty)$ such that the operator \tilde{B}_θ in (5.5) is unitarily equivalent to the operator Λ_σ in (5.7). It is easily seen that if the mapping U in (5.15) has a property that symmetric operators \tilde{B} in (5.4) and Λ_σ^0 in (5.8) are also unitarily equivalent w.r.t. U , then the unitary operator appearing in the proof of Theorem 5.5 coincides with the one defined by the formulas (5.15). Indeed, assuming that there is another mapping \tilde{U} provided by Theorem 3.2 on unitary equivalence for the systems Θ_Λ and Θ we would violate the uniqueness of the operator U , and thus $\tilde{U} = U$.

THEOREM 5.7. *Let $V(z) \in SL_0(\mathbb{R})$ satisfy the conditions of theorem 5.5. Then the operator T_h in the conclusion of the theorem 5.5 is accretive if and only if*

$$\gamma^2 + \gamma \int_0^\infty \frac{d\sigma(t)}{t} + 1 \geq 0. \quad (5.16)$$

The operator T_h is α -sectorial for some $\alpha \in (0, \pi/2)$ if and only if the inequality (5.16) is strict. In this case the exact value of angle α can be calculated by the formula

$$\tan \alpha = \frac{\int_0^\infty \frac{d\sigma(t)}{t}}{\gamma^2 + \gamma \int_0^\infty \frac{d\sigma(t)}{t} + 1}. \quad (5.17)$$

Proof. It was shown in [26] that for the system Θ in (5.13) described in the previous theorem the operator T_h is accretive if and only if the function

$$\begin{aligned} V_h(z) &= -i[W_\Theta^{-1}(-1)W_\Theta(z) + I]^{-1}[W_\Theta^{-1}(-1)W_\Theta(z) - I] \\ &= -i \frac{1 - [(m_\infty(z) + \bar{h})/(m_\infty(z) + h)][(m_\infty(-1) + h)/(m_\infty(-1) + \bar{h})]}{1 + [(m_\infty(z) + \bar{h})/(m_\infty(z) + h)][(m_\infty(-1) + h)/(m_\infty(-1) + \bar{h})]}, \end{aligned} \quad (5.18)$$

is holomorphic in $\text{Ext}[0, +\infty)$ and satisfies the following inequality

$$1 + V_h(0) V_h(-\infty) \geq 0. \quad (5.19)$$

Here $W_\Theta(z)$ is the transfer function of (5.13). It is also shown in [26] that the operator T_h is α -sectorial for some $\alpha \in (0, \pi/2)$ if and only if the inequality (5.19) is strict while the exact value of angle α can be calculated by the formula

$$\cot \alpha = \frac{1 + V_h(0) V_h(-\infty)}{|V_h(-\infty) - V_h(0)|}. \quad (5.20)$$

According to theorem 5.5 and equation (3.5)

$$W_{\Theta}(z) = (I - iV(z)J)(I + iV(z)J)^{-1}.$$

By direct calculations one obtains

$$W_{\Theta}(-1) = \frac{1 - i \left[\gamma + \int_0^{\infty} \frac{d\sigma(t)}{t+1} \right]}{1 + i \left[\gamma + \int_0^{\infty} \frac{d\sigma(t)}{t+1} \right]}, \quad W_{\Theta}^{-1}(-1) = \frac{1 + i \left[\gamma + \int_0^{\infty} \frac{d\sigma(t)}{t+1} \right]}{1 - i \left[\gamma + \int_0^{\infty} \frac{d\sigma(t)}{t+1} \right]}. \quad (5.21)$$

Using the following notations

$$a = \gamma + \int_0^{\infty} \frac{d\sigma(t)}{t+1} \quad \text{and} \quad b = \gamma + \int_0^{\infty} \frac{d\sigma(t)}{t},$$

and performing straightforward calculations we obtain

$$V_h(0) = \frac{a - b}{1 + ab} \quad \text{and} \quad V_h(-\infty) = \frac{a - \gamma}{1 + a\gamma}. \quad (5.22)$$

Substituting (5.22) into (5.20) and performing the necessary steps we get

$$\cot \alpha = \frac{1 + b\gamma}{b - \gamma} = \frac{\gamma^2 + \gamma \int_0^{\infty} \frac{d\sigma(t)}{t} + 1}{\int_0^{\infty} \frac{d\sigma(t)}{t}}. \quad (5.23)$$

Taking into account that $b - \gamma > 0$ we combine (5.19), (5.20) with (5.23) and this completes the proof of the theorem. \square

COROLLARY 5.8. *Let $V(z) \in SL_0(\mathbb{R})$ satisfy the conditions of theorem 5.5. Then the operator T_h in the conclusion of theorem 5.5 is accretive if and only if*

$$1 + V(0)V(-\infty) \geq 0. \quad (5.24)$$

The operator T_h is α -sectorial for some $\alpha \in (0, \pi/2)$ if and only if the inequality (5.24) is strict. In this case the exact value of angle α can be calculated by the formula

$$\tan \alpha = \frac{|V(-\infty) - V(0)|}{1 + V(0)V(-\infty)}. \quad (5.25)$$

Proof. Taking into account that

$$V(0) = \gamma + \int_0^{\infty} \frac{d\sigma(t)}{t},$$

$V(z) = V_{\Theta}(z)$, and $V_{\Theta}(-\infty) = \gamma$, we use (5.16) and (5.17) to obtain (5.24) and (5.25). \square

THEOREM 5.9. *Let $V(z) \in S_0(\mathbb{R})$ and satisfy the conditions of theorem 5.5. Then the system Θ of the form (5.13) is accretive and its symmetric operator A of the form (2.4) is such that its Kreĭn-von Neumann extension A_K of the form (2.8) does not coincide with its Friedrichs extension A_F of the form (2.9).*

Proof. The proof of the fact that Θ is accretive directly follows from the theorems 4.2 and 5.5. The second part follows from the theorem in [25] that states that a positive symmetric operator A admits a non-self-adjoint accretive extension T if and only if A_F and A_K do not coincide. \square

Below we will derive the formulas for calculation of the boundary parameter h in the restored Schrödinger operator T_h of the form (5.1). We consider two major cases.

Case 1. In the first case we assume that $\int_0^\infty \frac{d\sigma(t)}{t} < \infty$. This means that our function $V(z)$ belongs to the class $SL_{01}^K(\mathbb{R})$. In what follows we denote

$$b = \int_0^\infty \frac{d\sigma(t)}{t} \quad \text{and} \quad m = m_\infty(-0).$$

Suppose that $b \geq 2$. Then the quadratic inequality (5.16) implies that for all γ such that

$$\gamma \in \left(-\infty, \frac{-b - \sqrt{b^2 - 4}}{2} \right] \cup \left[\frac{-b + \sqrt{b^2 - 4}}{2}, +\infty \right) \quad (5.26)$$

the restored operator T_h is accretive. Clearly, this operator is extremal accretive if

$$\gamma = \frac{-b \pm \sqrt{b^2 - 4}}{2}.$$

In particular if $b = 2$ then $\gamma = -1$ and the function

$$V(z) = -1 + \int_0^\infty \frac{d\sigma(t)}{t - z}$$

is realized using an extremal accretive T_h .

Now suppose that $0 < b < 2$. For every $\gamma \in (-\infty, +\infty)$ the restored operator T_h will be accretive and α -sectorial for some $\alpha \in (0, \pi/2)$. Consider a function $V(z)$ defined by (5.9). Conducting realizations of $V(z)$ by operators T_h for different values of $\gamma \in (-\infty, +\infty)$ we notice that the operator T_h with the largest angle of sectoriality occurs when

$$\gamma = -\frac{b}{2}, \quad (5.27)$$

and is found according to the formula

$$\alpha = \arctan \frac{b}{1 - b^2/4}. \quad (5.28)$$

This follows from the formula (5.17), the fact that $\gamma^2 + \gamma b + 1 > 0$ for all γ , and the formula

$$\gamma^2 + \gamma b + 1 = \left(\gamma + \frac{b}{2} \right)^2 + \left(1 - \frac{b^2}{4} \right).$$

Now we will focus on the description of the parameter h in the restored operator T_h .

It was shown in [6] that the quasi-kernel \hat{A} of the realizing system Θ from theorem 5.5 takes a form

$$\begin{cases} \hat{A}y = -y'' + qy \\ y'(a) = \eta y(a) \end{cases}, \quad \eta = \frac{\mu \operatorname{Re} h - |h|^2}{\mu - \operatorname{Re} h} \quad (5.29)$$

On the other hand, since $\sigma(t)$ is also the distribution function of the positive self-adjoint operator, we can conclude that \hat{A} equals to the operator \tilde{B}_θ of the form (5.5). This connection allows us to obtain

$$\theta = \eta = \frac{\mu \operatorname{Re} h - |h|^2}{\mu - \operatorname{Re} h}. \quad (5.30)$$

Assuming that

$$h = x + iy$$

we will use (5.30) to derive the formulas for x and y in terms of γ . First, to eliminate parameter μ , we notice that (3.16) and (3.5) imply

$$W_\Theta(\lambda) = \frac{\mu - h}{\mu - \bar{h}} \frac{m_\infty(\lambda) + \bar{h}}{m_\infty(\lambda) + h} = \frac{1 - iV(z)}{1 + iV(z)}. \quad (5.31)$$

Passing to the limit in (5.31) when $\lambda \rightarrow -\infty$ and taking into account that $V(-\infty) = \gamma$ and $m_\infty(-\infty) = \infty$ we obtain²

$$\frac{\mu - h}{\mu - \bar{h}} = \frac{1 - i\gamma}{1 + i\gamma}.$$

Let us denote

$$a = \frac{1 - i\gamma}{1 + i\gamma}. \quad (5.32)$$

Solving (5.32) for μ yields

$$\mu = \frac{h - a\bar{h}}{1 - a}.$$

Substituting this value into (5.30) after simplification produces

$$\frac{x + iy - a(x - iy)x - (x^2 + y^2)(1 - a)}{x + iy - a(x - iy) - x(1 - a)} = \theta.$$

After straightforward calculations targeting to represent numerator and denominator of the last equation in standard form one obtains the following relation

$$x - \gamma y = \theta. \quad (5.33)$$

²The fact that $m_\infty(-\infty) = \infty$ (with the assumption $m_\infty(0) < \infty$ considered in this paper for the corresponding nonnegative Schrödinger operator) follows from (2.6), (3.17), (3.12), and (5.23) when $\operatorname{Re} h = -m_\infty(0)$, $\mu = \infty$ as well as from

$$V_{\Theta(\lambda)} = \left((\operatorname{Re} \mathbb{A} - \lambda I)^{-1} g, g \right) = \frac{\operatorname{Im} h}{m_\infty(\lambda) - m_\infty(0)}$$

and the relation $V_\Theta(-\infty) = 0$ established in [6], [15].

It was shown in [26] that the α -sectoriality of the operator T_h and (5.20) lead to

$$\tan \alpha = \frac{\operatorname{Im} h}{\operatorname{Re} h + m_\infty(-0)} = \frac{y}{x + m_\infty(-0)}. \tag{5.34}$$

Combining (5.33) and (5.34) one obtains

$$x - \gamma(x \tan \alpha + m_\infty(-0) \tan \alpha) = \theta,$$

or

$$x = \frac{\theta + \gamma m_\infty(-0) \tan \alpha}{1 - \gamma \tan \alpha}.$$

But $\tan \alpha$ is also determined by (5.17). Direct substitution of

$$\tan \alpha = \frac{b}{1 + \gamma(\gamma + b)}$$

into the above equation yields

$$x = \theta + \frac{[\theta + m_\infty(-0)]b\gamma}{1 + \gamma^2}.$$

Using the short notation and finalizing calculations we get

$$h = x + iy, \quad x = \theta + \frac{\gamma[\theta + m]b}{1 + \gamma^2}, \quad y = \frac{[\theta + m]b}{1 + \gamma^2}. \tag{5.35}$$

At this point we can use (5.35) to provide analytical and graphical interpretation of the parameter h in the restored operator T_h . Let

$$c = (\theta + m)b.$$

Again we consider three subcases.

Subcase 1: $b > 2$ Using basic algebra we transform (5.35) into

$$(x - \theta)^2 + \left(y - \frac{c}{2}\right)^2 = \frac{c^2}{4}. \tag{5.36}$$

Since in this case the parameter γ belongs to the interval in (5.26), we can see that h traces the highlighted part of the circle on the figure 1 as γ moves from $-\infty$ towards $+\infty$.

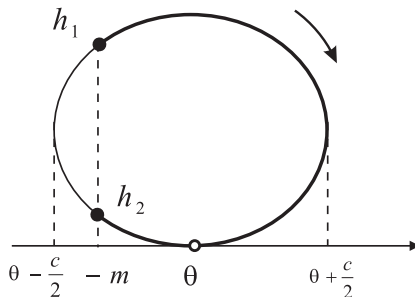


Figure 1. $b > 2$

We also notice that the removed point $(\theta, 0)$ corresponds to the value of $\gamma = \pm\infty$ while the points h_1 and h_2 correspond to the values $\gamma_1 = \frac{-b-\sqrt{b^2-4}}{2}$ and $\gamma_2 = \frac{-b+\sqrt{b^2-4}}{2}$, respectively (see figure 2.).

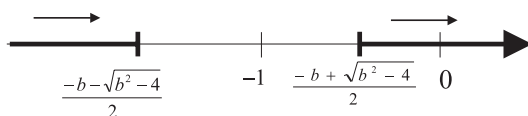


Figure 2. γ interval

Subcase 2: $b < 2$ For every $\gamma \in (-\infty, +\infty)$ the restored operator T_h will be accretive and α -sectorial for some $\alpha \in (0, \pi/2)$. As we have mentioned above, the operator T_h achieves the largest angle of sectorialilty when $\gamma = -\frac{b}{2}$. In this particular case (5.35) becomes

$$h = x + iy, \quad x = \frac{\theta(4 - b^2) - 2b^2m}{4 + b^2}, \quad y = \frac{4(\theta + m)b}{4 + b^2}. \tag{5.37}$$

The value of h from (5.37) is marked on the figure 3.

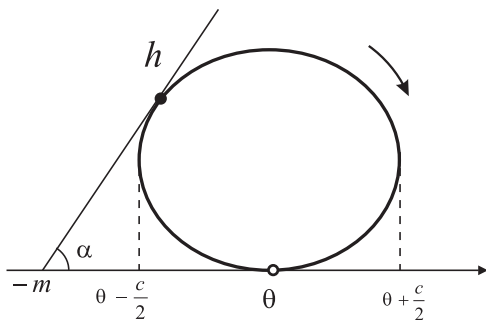


Figure 3. $b < 2$

Subcase 3: $b = 2$ The behavior of parameter h in this case is depicted on the figure 4. It shows that in this case the function $V(z)$ can be realized using an extremal accretive T_h when $\gamma = -1$. The value of the parameter h according to (5.35) then becomes

$$h = x + iy, \quad x = -m, \quad y = \theta + m. \tag{5.38}$$

Clockwise direction of the circle again corresponds to the change of γ from $-\infty$ to $+\infty$ and the marked value of h occurs when $\gamma = -1$.

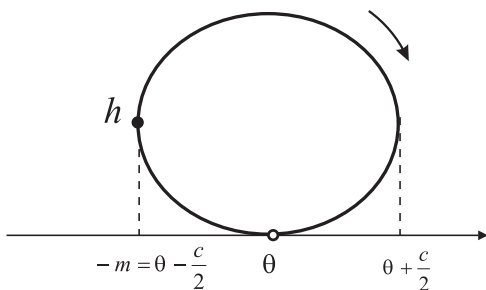


Figure 4. $b = 2$

Now we consider the second case.

Case 2. Here we assume that $\int_0^\infty \frac{d\sigma(t)}{t} = \infty$. This means that our function $V(z)$ belongs to the class $SL_0^K(\mathbb{R})$ and $b = \infty$. According to theorem 5.7 and formulas (5.16) and (5.17), the restored operator T_h is accretive if and only if

$$\gamma \geq 0,$$

and α -sectorial if and only if $\gamma > 0$. It directly follows from (5.17) that the exact value of the angle α is then found from

$$\tan \alpha = \frac{1}{\gamma}. \tag{5.39}$$

The latter implies that the restored operator T_h is extremal if $\gamma = 0$. This means that a function $V(z) \in SL_0^K(\mathbb{R})$ is realized by a system with an extremal operator T_h if and only if

$$V(z) = \int_0^\infty \frac{d\sigma(t)}{t - z}. \tag{5.40}$$

On the other hand since $\gamma \geq 0$ the function $V(z)$ is a Stieltjes function of the class $S_0(\mathbb{R})$. Applying realization theorems from [15] we conclude that $V(z)$ admits realization by an accretive system Θ of the form (3.1) with \mathbb{A}_R containing the Krein-von Neumann extension A_K as a quasi-kernel. Here A_K is defined by (2.8). This yields

$$\theta = -m_\infty(-0) = -m. \tag{5.41}$$

As in the beginning of the previous case we derive the formulas for x and y , where $h = x + iy$. Using (5.30) and (5.33) leads to

$$\begin{cases} \theta = \frac{\mu x - (x^2 + y^2)}{\mu - x}, \\ x = \theta + \gamma y. \end{cases} \tag{5.42}$$

Solving this system for x and y leads to

$$x = \frac{\theta + \mu\gamma^2}{1 + \gamma^2}, \quad y = \frac{(\mu - \theta)\gamma}{1 + \gamma^2}. \tag{5.43}$$

Combining (5.42) and (5.43) gives

$$x = \frac{-m + \mu\gamma^2}{1 + \gamma^2}, \quad y = \frac{(m + \mu)\gamma}{1 + \gamma^2}. \quad (5.44)$$

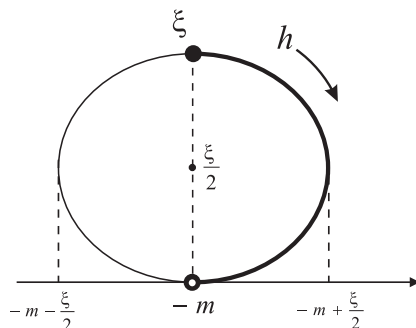


Figure 5. $b = \infty$

To proceed, we first notice that our function $V(z)$ satisfies the conditions of theorem 4.8 of [6]. Indeed, the inequality

$$\mu \geq \frac{(\operatorname{Im} h)^2}{m_\infty(-0) + \operatorname{Re} h} + \operatorname{Re} h,$$

turns into

$$\mu = \frac{y^2}{x - m} + x,$$

if you use $\theta = -m$ and the first equation in (5.42). Applying theorem 4.8 of [6] yields

$$\int_0^\infty \frac{d\sigma(t)}{1 + t^2} = \frac{\operatorname{Im} h}{|\mu - h|^2} \left(\sup_{y \in D(A_K)} \frac{|\mu y(a) - y'(a)|}{\left(\int_a^\infty (|y(x)|^2 + |l(y)|^2) dx \right)^{\frac{1}{2}}} \right)^2. \quad (5.45)$$

Taking into account that

$$\mu y(a) - y'(a) = (\mu + m)y(a)$$

and setting

$$c^{1/2} = \sup_{y \in D(A_K)} \frac{|y(a)|}{\left(\int_a^\infty (|y(x)|^2 + |l(y)|^2) dx \right)^{\frac{1}{2}}}, \quad (5.46)$$

we obtain

$$\frac{\operatorname{Im} h}{|\mu - h|^2} (\mu + m)^2 c = \int_0^\infty \frac{d\sigma(t)}{1 + t^2}. \quad (5.47)$$

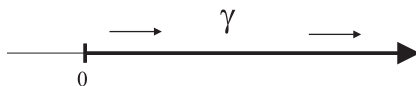


Figure 6. $\gamma \geq 0$

Considering that $\text{Im } h = y$ and combining (5.47) with (5.44) we use straightforward calculations to get

$$\mu = -m + \left(\frac{1}{\gamma c}\right) \int_0^\infty \frac{d\sigma(t)}{1+t^2}.$$

Let

$$\xi = \frac{1}{c} \int_0^\infty \frac{d\sigma(t)}{1+t^2}. \tag{5.48}$$

Then the last equation becomes

$$\mu = -m + \frac{\xi}{\gamma}. \tag{5.49}$$

Applying (5.49) on (5.44) yields

$$x = -m + \frac{\gamma \xi}{1 + \gamma^2}, \quad y = \frac{\xi}{1 + \gamma^2}, \quad \gamma \geq 0. \tag{5.50}$$

Following the previous case approach we transform (5.50) into

$$(x + m)^2 + \left(y - \frac{\xi}{2}\right)^2 = \frac{\xi^2}{4}. \tag{5.51}$$

The connection between the parameters γ and h in the accretive restored operator T_h is depicted in figures 5. and 6. As we can see h traces the highlighted part of the circle clockwise on the figure 5 as γ moves from 0 towards $+\infty$.

As we mentioned earlier the restored operator T_h is extremal if $\gamma = 0$. In this case formulas (5.50) become

$$x = -m, \quad y = \xi, \quad \gamma = 0, \tag{5.52}$$

where ξ is defined by (5.48).

6. Realizing systems with Schrödinger operator

Now once we described all the possible outcomes for the restored accretive operator T_h , we can concentrate on the main operator \mathbb{A} of the system (5.13). We recall that \mathbb{A} is defined by formulas (2.6) and beside the parameter h above contains also parameter μ . We will obtain the behavior of μ in terms of the components of our function $V(z)$ the same way we treated the parameter h . As before we consider two major cases dividing them into subcases when necessary.

Case 1. Assume that $b = \int_0^\infty \frac{d\sigma(t)}{t} < \infty$. In this case our function $V(z)$ belongs to the class $SL_{01}^K(R)$. First we will obtain the representation of μ in terms of x and y , where $h = x + iy$. We recall that

$$\mu = \frac{h - a\bar{h}}{1 - a},$$

where a is defined by (5.32). By direct computations we derive that

$$a = \frac{1 - \gamma^2}{1 + \gamma^2} - \frac{2\gamma}{1 + \gamma^2}i, \quad 1 - a = \frac{2\gamma^2}{1 + \gamma^2} + \frac{2\gamma}{1 + \gamma^2}i,$$

and

$$h - a\bar{h} = \left(\frac{2\gamma^2}{1 + \gamma^2}x + \frac{2\gamma}{1 + \gamma^2}y \right) + \left(\frac{2}{1 + \gamma^2}y + \frac{2\gamma}{1 + \gamma^2}x \right)i.$$

Plugging the last two equations into the formula for μ above and simplifying we obtain

$$\mu = x + \frac{1}{\gamma}y. \tag{6.1}$$

We recall that during the present case x and y parts of h are described by the formulas (5.35).

Once again we elaborate in three subcases.

Subcase 1: $b > 2$ As we have shown this above, the formulas (5.35) can be transformed into equation of the circle (5.36). In this case the parameter γ belongs to the interval in (5.26), the accretive operator T_h corresponds to the values of h shown in the bold part of the circle on the figure 1 as γ moves from $-\infty$ towards $+\infty$.

Substituting the expressions for x and y from (5.35) into (6.1) and simplifying we get

$$\mu = \theta + \frac{(\theta + m)b}{\gamma}. \tag{6.2}$$

The connection between values of γ and μ is depicted on the figure 7.

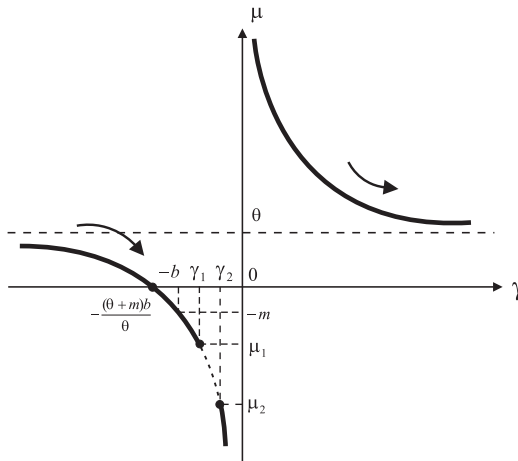


Figure 7. $b > 2$

We note that $\mu = 0$ when $\gamma = -\frac{(\theta+m)b}{\theta}$. Also, the endpoints

$$\gamma_1 = \frac{-b - \sqrt{b^2 - 4}}{2} \quad \text{and} \quad \gamma_2 = \frac{-b + \sqrt{b^2 - 4}}{2}$$

of γ -interval (5.26) are responsible for the μ -values

$$\mu_1 = \theta + \frac{(\theta + m)b}{\gamma_1} \quad \text{and} \quad \mu_2 = \theta + \frac{(\theta + m)b}{\gamma_2}.$$

The values of μ that are acceptable parameters of operator \mathbb{A} of the restored system make the bold part of the hyperbola on the figure 7. It follows from theorem 4.2 that the operator \mathbb{A} of the form (2.6) is accretive if and only if $\gamma \geq 0$ and thus μ sweeps the right branch on the hyperbola. We note that figure 7 shows the case when $-m < 0$, $\theta > 0$, and $\theta > -m$. Other possible cases, such as $(-m < 0, \theta < 0, \theta > -m)$, $(-m < 0, \theta = 0)$, and $(m = 0, \theta > 0)$ require corresponding adjustments to the graph shown in the picture 7.

Subcase 2: $b < 2$ For every $\gamma \in (-\infty, +\infty)$ the restored operator T_h will be accretive and α -sectorial for some $\alpha \in (0, \pi/2)$. As we have mentioned above, the operator T_h achieves the largest angle of sectorialilty when $\gamma = -\frac{b}{2}$. In this particular case (5.35) becomes

$$h = x + iy, \quad x = \frac{\theta(4 - b^2) - 2b^2m}{4 + b^2}, \quad y = \frac{4(\theta + m)b}{4 + b^2}.$$

Substituting $\gamma = b/2$ into (6.1) we obtain

$$\mu = -(\theta + 2m). \tag{6.3}$$

This value of μ from (6.3) is marked on the figure 8. The corresponding operator \mathbb{A} of the realizing system is based on these values of parameters h and μ .

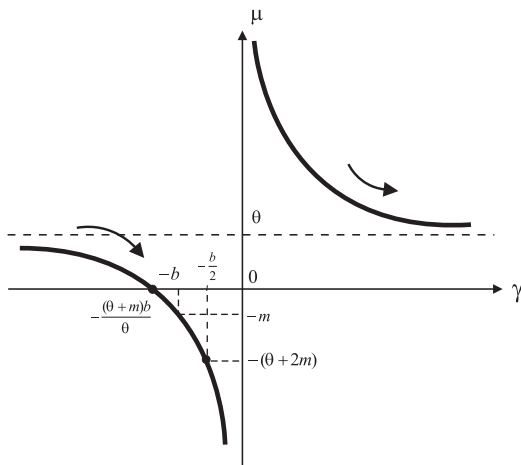


Figure 8. $b < 2$ and $b = 2$

Subcase 3: $b = 2$ The behavior of parameter μ in this case is also shown on the figure 8. It was shown above that in this case the function $V(z)$ can be realized using an extremal accretive T_h when $\gamma = -1$. The values of the parameters h and μ then become

$$h = x + iy, \quad x = -m, \quad y = \theta + m, \quad \mu = -(\theta + 2m).$$

The value of μ above is marked on the left branch of the hyperbola and occurs when $\gamma = -1 = -b/2$.

Case 2. Again we assume that $\int_0^\infty \frac{d\sigma(t)}{t} = \infty$. Hence $V(z) \in SL_0^K(R)$ and $b = \infty$. As we mentioned above the restored operator T_h is accretive if and only if $\gamma \geq 0$ and α -sectorial if and only if $\gamma > 0$. It is extremal if $\gamma = 0$. The values of x , y , and μ were already calculated and are given in (5.50) and (5.49), respectively. That is

$$x = -m + \frac{\gamma \xi}{1 + \gamma^2}, \quad y = \frac{\xi}{1 + \gamma^2}, \quad \mu = -m + \frac{\xi}{\gamma}, \quad \gamma \geq 0.$$

where ξ is defined in (5.48). Figure 9. gives graphical representation of this case. Only the right bold branch of hyperbola shows the values of μ in the case $b = \infty$. If $m = 0$ then

$$\mu = \frac{\xi}{\gamma}$$

and the graph should be adjusted accordingly.

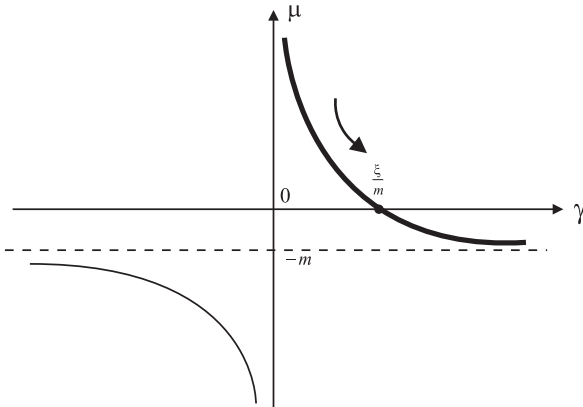


Figure 9. $b = \infty$

In the case when $\gamma = 0$ and T_h is extremal we have

$$x = -m, \quad y = \xi, \quad \mu = \infty, \quad h = -m + i\xi, \tag{6.4}$$

and according to (2.6) we have

$$\Delta y = -y'' + q(x)y + [(-m + i\xi)y(a) - y'(a)]\delta(x - a), \tag{6.5}$$

that is the main operator of the realizing system.

Example

We conclude this paper with simple illustration. Consider a function

$$V(z) = \frac{i}{\sqrt{z}}. \quad (6.6)$$

A direct check confirms that $V(z)$ is a Stieltjes function. It was shown in [23] (see pp. 140-142) that the inversion formula

$$\sigma(\lambda) = C + \lim_{y \rightarrow 0} \frac{1}{\pi} \int_0^\lambda \operatorname{Im} \left(\frac{i}{\sqrt{x+iy}} \right) dx \quad (6.7)$$

describes the distribution function for a self-adjoint operator

$$\begin{cases} \tilde{B}_0 y = -y'' \\ y'(0) = 0. \end{cases}$$

The corresponding to \tilde{B}_0 symmetric operator is

$$\begin{cases} B_0 y = -y'' \\ y(0) = y'(0) = 0. \end{cases} \quad (6.8)$$

It was also shown in [23] that $\sigma(\lambda) = 0$ for $\lambda \leq 0$ and

$$\sigma'(\lambda) = \frac{1}{\pi\sqrt{\lambda}} \quad \text{for } \lambda > 0. \quad (6.9)$$

By direct calculations one can confirm that

$$V(z) = \int_0^\infty \frac{d\sigma(t)}{t-z} = \frac{i}{\sqrt{z}},$$

and that

$$\int_0^\infty \frac{d\sigma(t)}{t} = \int_0^\infty \frac{dt}{\pi t^{3/2}} = \infty.$$

It is also clear that the constant term in the integral representation (4.1) is zero, i.e. $\gamma = 0$.

Let us assume that $\sigma(t)$ satisfies our definition of spectral distribution function of the pair B_0, \tilde{B}_0 given in the section 5. Operating under this assumption, we proceed to restore parameters h and μ and apply formulas (5.50) for the values $\gamma = 0$ and $m = -\theta = 0$. This yields $x = 0$. To obtain y we first find the value of

$$\int_0^\infty \frac{d\sigma(t)}{1+t^2} = \frac{1}{\sqrt{2}},$$

and then use formula (5.46) to get the value of c . This yields $c = 1/\sqrt{2}$. Consequently,

$$\xi = \frac{1}{c} \int_0^\infty \frac{d\sigma(t)}{1+t^2} = 1,$$

and hence $h = yi = i$. From (5.49) we have that $\mu = \infty$ and (6.5) becomes

$$\mathbb{A}y = -y'' + [iy(0) - y'(0)]\delta(x). \tag{6.10}$$

The operator T_h in this case is

$$\begin{cases} T_h y = -y'' \\ y'(0) = iy(0). \end{cases}$$

The channel vector g of the form (3.12) then equals

$$g = \delta(x), \tag{6.11}$$

satisfying

$$\text{Im } \mathbb{A} = \frac{\mathbb{A} - \mathbb{A}^*}{2i} = KK^* = (\cdot, g)g,$$

and channel operator $Kc = cg, (c \in \mathbb{C})$ with

$$K^*y = (y, g) = y(0). \tag{6.12}$$

The real part of \mathbb{A}

$$\text{Re } \mathbb{A}y = -y'' - y'(0)\delta(x)$$

contains the self-adjoint quasi-kernel

$$\begin{cases} \widehat{A}y = -y'' \\ y'(0) = 0. \end{cases}$$

A system of the Livšic type with Schrödinger operator of the form (5.13) that realizes $V(z)$ can now be written as

$$\Theta = \left(\begin{array}{cc} \mathbb{A} & K \quad 1 \\ \mathcal{H}_+ \subset L_2[a, +\infty) \subset \mathcal{H}_- & \mathbb{C} \end{array} \right).$$

where \mathbb{A} and K are defined above. Now we can back up our assumption on $\sigma(t)$ to be the spectral distribution function of the pair B_0, \tilde{B}_0 . Indeed, calculating the function $V_\Theta(z)$ for the system Θ above directly via formula (3.17) with $\mu = \infty$ and comparing the result to $V(z)$ gives the exact value of $h = i$. Using the reasoning of remark 5.6 we confirm that $\sigma(t)$ is the spectral distribution function of the pair B_0, \tilde{B}_0 .

REMARK 6.1. All the derivations above can be repeated for a Stieltjes like function

$$V(z) = \gamma + \frac{i}{\sqrt{z}}, \quad -\infty < \gamma < +\infty, \gamma \neq 0$$

with very minor changes. In this case the restored values for h and μ are described as follows:

$$h = x + iy, \quad x = \frac{\gamma}{1 + \gamma^2}, \quad y = \frac{1}{1 + \gamma^2}, \quad \mu = \frac{1}{\gamma}.$$

The dynamics of changing h according to changing γ is depicted on the figure 5 where the circle has the center at the point $i/2$ and radius of $1/2$. The behavior of μ is described by a hyperbola $\mu = 1/\gamma$ (see figure 9 with $\theta = 0$). In the case when $\gamma > 0$ our function becomes Stieltjes and the restored system Θ is accretive. The operators \mathbb{A} and K of the restored system are given according to the formulas (2.6) and (3.14), respectively.

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