TEST 1

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MTH 6620 | S.S.N.____________________

1. Decide whether each of the given set is a group with respect to the indicated operation. If it is not a group, state a condition that fails to hold.

   (a) The set of all positive rational numbers with operation multiplication

   (b) The set of all integers $\mathbb{Z}$ with the binary operation $*$ defined on $\mathbb{Z}$ by the rule

$$x * y = xy + y.$$

   (c) the set of all matrices in $M_3(\mathbb{R})$ that have the form

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

   (all three numbers a, b, and c are nonzero) with respect to the matrix multiplication.

2. Let $\mathbb{Z}_n = \{[0],[1],...[n-1]\}$ be the set of the congruence classes mod n. Let G be the group $M_2(\mathbb{Z}_5)$ under addition. (*Note:* $M_2(\mathbb{Z}_5)$ consists of (2×2) matrices whose entries belong to $\mathbb{Z}_5$). List all the elements of the cyclic subgroup $\langle A \rangle$ of G where

$$A = \begin{bmatrix} [0] & [1] \\ [2] & [4] \end{bmatrix}.$$

3. Show that the set of the nonzero elements of $\mathbb{Z}_7$ (see the definition of $\mathbb{Z}_n$ in the Problem 2) is a cyclic group.

4. Show that if the group G has three elements, it must be abelian.

5. Let G be a group. If $a \in G$, define $N(a) = \{x \in G \mid xa = ax\}$. $N(a)$ is usually called the *normalizer* of a in G. Show that $N(a)$ is a subgroup of G.

6. If G is an abelian group and the group G' is a homomorphic image of G, prove that G' is abelian.

7. Prove that any infinite cyclic group is isomorphic to $\mathbb{Z}$ (group of all integers) under addition.

8. The *center* $Z(G)$ of a group G is defined as

$$Z(G) = \{z \in G \mid zx = xz \text{ for all } x \in G\}.$$

Prove that $Z(G)$ is a normal subgroup of G.

9. Let G be a group. If N is a normal subgroup of G and $a \in G$ is of order $o(a)$, prove that the order, m, of Na in G/N is a divisor of $o(a)$.

10. For $G = S_3$ prove that $G \approx \mathcal{J}(G)$, where $\mathcal{J}(G)$ is *the group of inner automorphisms* of G.